Physics 690/482-02, Spring 2005 Josh Erlich

Problem Set 4: Electrodynamics in $d+1$ dimensions

1. *Maxwell's equations from Maxwell's equations*

In any number of dimensions the covariant form of Maxwell's equations is:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -4\pi J^\nu \\ \epsilon^{\mu_1\mu_2\cdots\mu_{d+1}}\partial_{\mu_1} F_{\mu_2\mu_3} &= 0 \end{aligned}$$

- In 3+1 dimensions, show that the covariant form of Maxwell's equations is equivalent to the usual 4 Maxwell's equations in terms of $\mathbf{E}$ and $\mathbf{B}$.
- In $d+1$ dimensions, define $E_i = F_{i0}$ and $B_{i_1\cdots i_{d-2}} = \frac{1}{2(d-2)!}\epsilon_{i_1\cdots i_d}F^{i_{d-1}i_d}$. (A field with $d-2$ antisymmetric indices is called a $(d-2)$-form field.) What are Maxwell's equations in terms of E_i and $B_{i_1\cdots i_{d-2}}$?
- *Counting components*: How many independent components are there in an antisymmetric rank-2 tensor $F^{\mu\nu}$ in $d+1$ dimensions? Compare that to the number of independent components in E_i and $B_{i_1\cdots i_{d-2}}$. (*Hint:* How many ways can you choose $d-2$ different indices from d possible choices?)

2. *Longitudinal light*

Maxwell's equations describe $d+1$ dimensional *transverse* waves. Here you will consider the complementary theory, which has only *longitudinal* waves as solutions. The Lagrangian density for this theory is:

$$\mathcal{L} = \pm\frac{1}{2}\left((\partial_\mu A^\mu)^2 + m^2 A_\mu A^\mu\right).$$

- Derive the Euler-Lagrange equations for A^μ.
- Show that the solutions are longitudinal waves of mass m. (*Hint:* An ansatz for the solution is $A^\mu = \varepsilon^\mu e^{-ik\cdot x}$.)

- In $d+1$ dimensions, how many degrees of freedom are contained in A^μ?
- The canonical momenta for the spatial components of the vector field, $\frac{\partial \mathcal{L}}{\partial(\partial_0 A^i)}$, vanish. Show that the equations of motion allow you to determine A^i and $\partial_0 A^i$ if you know A^0 and $\Pi^0 \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A^0)}$. This is good because it means that a complete set of initial conditions is given by A^0 and Π^0, and it is not necessary to specify any additional canonical momenta at an initial time in order to specify a solution.
- Using the equations of motion, write the Hamiltonian,

  $$H = \int d^d x \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 A^0)} \partial_0 A^0 - \mathcal{L} \right],$$

  in terms of A^0 and Π^0. What sign must be chosen for the Lagrangian in order for the Hamiltonian to be bounded below?
- Compare the Hamiltonian in terms of A^0 and Π^0 with the Hamiltonian for the free massive scalar field.
- If $m = 0$, how does the field $A^0(r)$ vary with respect to the distance r from a point source in $d+1$ dimensions?