Physics 690/482-02 Josh Erlich

Problem Set 2: Geometry II

1. *Fun with coordinate transformations*

Prove that:

- If V^i is a contravariant vector, then $V_i \equiv g_{ij}V^j$ is a covariant vector.
- The area element $d^dx\sqrt{g}$ is a scalar. *Hint*: Consider the Jacobian that appears in the transformation of d^dx.
- If V_{ij} is a rank-2 covariant tensor, then $V_{ij}\, g^{ij}$ is a scalar.

2. *The Weyl tensor*

Einstein's equations involve contractions of the Riemann tensor with the metric (traces), but not the Riemann tensor itself. The Weyl tensor is defined as the *trace-free* part of the Riemann tensor with all lowered indices, *i.e* the part of the Riemann tensor for which all contractions with the metric vanish. It is possible for the Weyl tensor to be nonvanishing even in a vacuum solution to Einstein's equations.

- Show that any rank-2 covariant tensor can be written in the form $T_{ij} = t_{ij} + \alpha\, g_{ij}\, T^k_k$ where $t^i_i = 0$. What is α in d-dimensions?
- The Riemann tensor can be written as,

  $$R_{ijkl} = W_{ijkl} + \alpha(g_{ik}R_{lj} - g_{il}R_{kj} - g_{jk}R_{li} + g_{jl}R_{ki}) + \beta R(g_{ik}\, g_{jl} - g_{il}\, g_{jk})$$

  where all contractions of the Weyl tensor W_{ijkl} with the metric vanish. What are the coefficients α and β in d-dimensions?

3. *Kaluza-Klein modes*

Consider a 5+1 dimensional world compactified on a spacelike 2-torus of radii R_1, R_2 with metric,

$$ds^2 = -dt^2 + \sum_{i=1}^{3}(dx^i)^2 + R_1^2\, d\theta_1^2 + R_2^2\, d\theta_2^2.$$

Imagine a classical particle of mass m moving inertially (and relativistically) in this geometry. What is the dispersion relation for the particle in terms of the momenta in the six directions?

Imagine that the torus is much smaller than our measuring devices and ourselves, so that we can only measure motion in the directions $t, x_1, \ldots, x_3$. Thinking about the dispersion relation, what would be our interpretation of the components of momentum along the torus?

Now consider a relativistic scalar field $\phi(x)$ in the same spacetime satisfying the massive wave equation,

$$(-\partial_t^2 + \nabla^2 - m^2)\,\phi(x) = 0,$$

where ∇^2 is the spatial Laplacian. Write the wave equation in the coordinates defined by the metric above. Find plane wave solutions to the wave equation using separation of variables. What is the dispersion relation for this wave?

Compare the dispersion relations for the relativistic particle and the wave. What is the difference between the two cases?