Physics 690/482-02 Josh Erlich
Problem Set 1: Geometry

1. Consider a path along a two-dimensional cylinder of radius R. Write the metric in a convenient set of coordinates. Calculate the Christoffel symbols Γ^i_{jk}. Write down the two components of the geodesic equation on the cylinder and solve them. Draw a typical geodesic.

Now consider a four-dimensional cylinder of radius R. What are the four components of the geodesic equation? Describe the geodesics.

Write down all the components of the Riemann curvature tensor on the four-dimensional cylinder. (*Hint*: Think about it before you start calculating.)

2. Consider three-dimensional flat space in Euclidean coordinates (x^1, x^2, x^3) with metric $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. Transform to spherical coordinates,

$$\begin{aligned} x^1 &= r \sin\theta \cos\phi \\ x^2 &= r \sin\theta \sin\phi \\ x^3 &= r \cos\theta. \end{aligned}$$

Show that the metric in spherical coordinates is diagonal, with $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$.

A static electric field $\mathbf{E}$ can be expressed as the gradient of a potential, $\mathbf{E} = -\nabla\varphi(\mathbf{x})$. The field in the presence of a static charge density ρ satisfies $\nabla \cdot \mathbf{E} = 4\pi\rho$. Write this equation in terms of the potential φ in spherical coordinates.

3. Consider a three-dimensional metric of the form,

$$ds^2 = e^{-ky}\left(dx_1^2 + dx_2^2\right) + dy^2$$

Show that there is a choice of coordinates such that the metric can be written as,

$$ds^2 = f(z)\left(dx_1^2 + dx_2^2 + dz^2\right),$$

for some function $f(z)$. What is $f(z)$? Any geometry in which the metric can be written as a function times the flat metric is called *conformally flat.* We will call a set of coordinates in which this property is manifest *conformally flat coordinates.* Conformally flat metrics will play an important role in our study of extra dimensions.

Using the conformally flat form of the metric, calculate the length of the line along the z-axis ($x_1 = x_2 = 0$), stretching from $z = z_i$ to $z = z_f$. What is the result as $z_f \to \infty$?

What is the volume element in the conformally flat coordinates? Calculate the volume of the surface specified by $x_1^2 + x_2^2 = R^2$ between the slices $z = z_i$ and $z = z_f$. What is the result as $z_f \to \infty$?

In the original coordinates, calculate the length of the line and volume of the surface described as above, except that they now extend from $y = y_i$ to $y = y_f$. What happens as $y_f \to \infty$?