Physics 690/482-02 Spring, 2005 Josh Erlich

Notes: Massive Electrodynamics in a compact extra dimension

Consider a world in 4+1 dimensions, in which one of those dimensions is compactified on a circle of size R. Imagine that a massive vector field exists in this world.

Comparison to the scalar field on a circle

The massive vector field with an extra circle dimension has the same Lagrangian as in the case of noncompact dimensions:

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu} - \frac{m^2}{2}\, A_\mu A^\mu,$$

and hence also the same equations of motion,

$$\partial_\mu F^{\mu\nu} = m^2\, A^\nu,$$

and satisfies the Bianchi identity,

$$\epsilon^{\mu_0\cdots\mu_4}\partial_{\mu_2} F_{\mu_3\mu_4} = 0.$$

The difference between the noncompact and compact situations is in the boundary conditions. Just as we have discussed in the case of the scalar field, wavenumbers in compact dimensions take discrete values. In the case of the vector field, though, something more interesting happens.

Recall that in an extra circle dimension a scalar field of mass m has the interpretation of a tower of 3+1 dimensional scalar fields with masses,

$$m_n^2 = m^2 + \frac{n^2}{R^2}\,.$$

In the noncompact case there is one mode per wavenumber (*i.e.* one degree of freedom), and the same is true in the compact case (except that a component of the wavenumber is discrete).

In the case of the vector field, we might expect to find a tower of massive vector fields, but that can't be the whole story. Recall that the (transverse) massive vector field in $d+1$ noncompact dimensions has d degrees of freedom, *i.e.* d different solutions for each wavenumber. So a vector in d dimensions has only $d-1$ degrees of freedom. That means that if there is to be just a tower of d-dimensional vector fields, then we're one degree of freedom short from what we started with.

Just by counting degrees of freedom, then, it seems plausible that in a compact extra dimension the massive vector field should become a tower of massive vector fields *in addition to a tower of scalar fields.* Let's see how this works.

Plane waves on a circle

Recall that by differentiating the equations of motion we obtain the **Lorentz gauge condition**,

$$\partial_\nu A^\nu = 0,$$

so that the equation of motion becomes the Klein-Gordon equation for each component of A^ν:

$$\Box_{4+1} A^\nu = m^2\, A^\nu .$$

A complete set of solutions are plane waves:

$$A^\nu = \varepsilon^\nu \, \exp i \left[\omega t - \mathbf{k}\cdot\mathbf{x} - \frac{n}{R}\, y \right],$$

where by $\mathbf{k}$ and $\mathbf{x}$ we mean the three spatial components in the noncompact dimensions, and $y \in (0, 2\pi R)$ is the coordinate in the circle dimension.

The **polarization vector** ε^ν is independent of $\mathbf{x}$, and for the time being we won't normalize it. From the Klein-Gordon equation we obtain the dispersion relation just as for the scalar field,

$$\omega^2 - \mathbf{k}^2 = m^2 + \frac{n^2}{R^2}\, .$$

From the Lorentz gauge condition we find that the plane waves are transverse,

$$\varepsilon \cdot k = -\omega\, \varepsilon^0 + \varepsilon_i k^i + \varepsilon^y \, \frac{n}{R} = 0,$$

where the index i is summed over the noncompact spatial coordinates, $i = 1, 2, 3$.

The general solution to the equations of motion can be decomposed into the complete set of plane wave solutions. Hence,

$$A^\nu(t,\mathbf{x},y) = \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi R}\,\varepsilon_n^\nu(\mathbf{k})\,\exp i\left[\omega(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x} - \frac{n\,y}{R}\right],$$

where,

$$\omega(\mathbf{k})^2 = \mathbf{k}^2 + m^2 + \frac{n^2}{R^2},$$

and,

$$\varepsilon_n^\nu\,k_\nu = 0.$$

Decomposition into 3+1 dimensional fields

We can independently consider the modes with $\varepsilon^y = 0$ and the modes with $\varepsilon_i k^i = 0$ (in a particular coordinate frame, say the rest frame of the plane wave). Any solution can be written as a sum of such modes.

With $\varepsilon^y = 0$, the Klein-Gordon equation acting on the plane wave solutions take the form,

$$\Box_5 A^\nu = \Box_4 A^\nu - \frac{n^2}{R^2}\,A^\nu = m^2\,A^\nu\ , \quad \nu = 0,\ldots,3.$$

The Lorentz gauge condition for the modes with $\varepsilon^y = 0$ is,

$$\sum_{\mu=0}^{3} \partial_\mu A^\mu = 0.$$

These last two equations are the equations of motion for the massive vector field in $3+1$ dimensions. Hence, the plane wave solutions with $\varepsilon^y = 0$ are the same as those for a tower of massive vector fields in $3+1$ dimensions with masses,

$$m_n^2 = m^2 + \frac{n^2}{R^2}.$$

But now we can see where the extra degree of freedom went – it is just the solutions with $\varepsilon^i k_i = 0$ and $\varepsilon^y \neq 0$, which we have not considered yet. The vector field still satisfies the Klein-Gordon equation, but only the time-component and the y-component are nonvanishing.

From the 3+1 dimensional perspective, A^y is a scalar under Lorentz transformations, so the Lorentz gauge condition simply relates A^0 to this scalar. (Recall also from our discussion of the Hamiltonian that A^0 is not an independent field, anyway.) The Klein-Gordon equation for A^y is the equation of motion for a scalar field, so what we have found is that *the modes with $\epsilon_i k^i = 0$ are equivalent to a tower of scalar fields with the same masses as the tower of vector fields.*

Notice that in the rest frame, the scalar mode with $\varepsilon^i k_i = 0$ has $\varepsilon^\mu \propto k^\mu$ for $\mu = 0, 1, 2, 3$. Hence, such modes are sometimes thought of as longitudinal modes of the vector field, and A^y is sometimes referred to as the **longitudinal component** or **scalar component** of the vector field.

Coupling to sources

The equations of motion in the presence of a source are,

$$\partial_\mu F^{\mu\nu} = m^2 A^\nu - 4\pi J^\nu.$$

As a 3+1 dimensional observer, let's say you create a conserved current with $J^y = 0$, such that,

$$\partial_\mu J^\mu = \partial_t \rho + \nabla_3 \cdot \mathbf{J} = 0.$$

Then we could solve the equations of motion in the presence of the source, and we would find that we can consistently take $A^y = 0$ in the solution because $J^y = 0$. The solution for A^μ at large distances from the source (compared to R) will be approximately the $3+1$ dimensional solution. The **scalar component** A^y is not turned on by the source.

However, if we turn on a 4+1 dimensional conserved current with a nonvanishing component in the circle direction, *i.e.*

$$\partial_\mu J^\mu = \partial_t \rho + \nabla_3 \cdot \mathbf{J} + \partial_y J^y = 0,$$

then to a 3+1 dimensional observer it would appear that the current is not conserved:

$$\sum_{\mu=0}^{3} \partial_\mu J^\mu = -\partial_y J^y.$$

In this case, because $J^y \neq 0$, the longitudinal component A^y is turned on by the source. From a $3+1$ dimensional perspective, we are learning that the longitudinal component only vanishes if the current is conserved. This is consistent with the interpretation of A^y as the longitudinal component of the vector field. Consider a 3+1 dimensional massive vector field. Taking a derivative of the equations of motion in the presence of a source, we have,

$$\partial_\nu \left(\partial_\mu F^{\mu\nu}\right) = \partial_\nu \left(m^2 \, A^\nu - 4\pi J^\nu\right) = 0.$$

If the current is conserved, so that $\partial_\nu J^\nu = 0$, then the solutions satisfy the Lorentz gauge condition, and hence also the condition for transverseness. If the current is not conserved, then there is no solution satisfying the Lorentz gauge condition, so that the longitudinal modes must be turned on in the solution.

We have not yet spoken about **gauge invariance**, so this is for Jackson experts (and for the rest of you to look back at later): One can show that if the current is made out of fields coupled to A^μ in a gauge invariant way, then the Lorentz gauge condition must be satisfied: $\partial_\mu A^\mu = 0$. As a result, by the equations of motion it follows that the current is conserved in such a theory: $\partial_\mu J^\mu = 0$. So it seems as though gauge invariance in a compact extra dimension does not imply gauge invariance from the lower dimensional perspective. We will revisit this point again.

Coming up next...

Our next topic will be massless electrodynamics in compact extra dimensions. We will have to face gauge invariance in all its glory, so I recommend that you remind yourself what gauge invariance is before then. Your favorite book on electrodynamics would serve just fine.