Physics 690/482-02 Spring, 2005 Josh Erlich

Notes: Electrodynamics in $d+1$ dimensions

Maxwell's remarkable theory

It is remarkable that James Clerk Maxwell's (1831-1879) theory of electrodynamics has survived the revolutions of special relativity and quantum mechanics. It was Maxwell who not only described the forces of electricity and magnetism, but also suggested that light, which was measured to propagate at roughly the same speed as electromagnetic signals, was in fact a manifestation of the electromagnetic field. Prior to the observation of the coincidence of these speeds there was no reason to suspect there was any relation between these seemingly different phenomena. The measured equivalence of these speeds was too much for Maxwell to accept as a coincidence, and the rest, as they say, is history.

It is interesting to note that the theory of electricity, magnetism and light is a wonderful example of the modern quest for unification and beauty in the laws of physics. Electricity and magnetism, which at first appear to be independent forces, are described by a single 4-vector field in 3+1 dimensions. Furthermore, Maxwell conjectured that light is a propagating wave of the same field because it was too unnatural to conceive that these things should all propagate at the same speed without there being some relation between them.

We will not discuss this further except perhaps much later in this course, but there is also a **duality** in Maxwell's equations: if you take $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$, the source-free Maxwell's equations are invariant. This duality in the description of the electromagnetic field is similar to the dualities often discussed in the context of string theory and supersymmetric gauge theories.

A modern derivation of Maxwell's theory

We will rewrite history and derive Maxwell's theory of electromagnetism from a modern perspective. We start with the assumptions that 1) the electromagnetic interaction is carried by fields, and 2) the theory is Lorentz invariant. Maxwell was not aware of the symmetry structure of his theory, so he could not have derived his theory the way we will.

We have already studied scalar fields, but we found that such fields are attractive between like charges so a collection of scalar fields can't be the right theory of electromagnetism.

The next simplest field transforming covariantly under Lorentz transformations is the vector field, A^μ. So we will derive the most general theory of a vector field. We proceed by constructing an action which satisfies three basic properties:

- Lorentz invariance
- No more than two derivatives
- At most quadratic in the fields

The first condition is obvious (although it was unknown to Maxwell), and means that the action should be a Lorentz scalar. The second condition implies that the field has ordinary kinetic terms so that a complete set of initial conditions requires knowledge of just the fields and their time derivatives, and not higher derivatives. (We will clarify this later when we discuss the Hamiltonian.) The last requirement is so that the Euler-Lagrange equations will be linear in the field, and is included in the list of requirements for simplicity. If we can't formulate an appropriate theory satisfying the last two of these requirements we can then try to relax these conditions.

A complete set of terms satisfying these requirements is:
No derivatives: $A^\mu A_\mu$
One derivative: None by Lorentz invariance (except in 3 dimensions)
Two derivatives: $\partial_\mu A^\nu\, \partial_\nu A^\mu$, $\partial_\mu A^\mu\, \partial_\nu A^\nu$, $\partial_\mu A^\nu\, \partial^\mu A_\nu$

The first two of the two-derivative terms listed above are equivalent up to addition of a total derivative, so we only need to keep one of those terms. (The action is an integral of the Lagrangian, so total derivatives don't contribute.)

Three dimensional electrodynamics is unique. In three dimensions the term $A_\mu \partial_\nu A_\rho\, \epsilon^{\mu\nu\rho}$ is Lorentz invariant and can be added to the Lagrangian. It is called the **Chern-Simons term**. Here, $\epsilon^{\mu\nu\rho}$ is the completely antisymmetric tensor with $\epsilon^{123} = +1$, and is invariant under Lorentz transformations connected to the identity.

In $d+1$ dimensions the analogous invariant tensor is $\epsilon^{\mu_1 \cdots \mu_{d+1}}$, defined with $\epsilon^{012\cdots d} = +1$. The components of $\epsilon^{\mu_1 \cdots \mu_{d+1}}$ vanish except when all the indices take different values. Lowering all the indices with the Minkowski metric, we obtain $\epsilon_{012\cdots d} = -1$. There are terms analogous to the Chern-Simons term in any odd dimension (≥ 3), but in more than three dimensions there are more derivatives and more than two factors of the field.

The Chern-Simons term is a total derivative, yet it has the interesting consequence of contributing topologically to the photon mass. We will not study the details of this interesting theory any further in this course.

Back to the generic theory, we can normalize A^μ to fix any one of the terms in the Lagrangian density (up to a sign). Hence, the generic Lagrangian satisfying our requirements is:

$$\mathcal{L} = \pm\frac{1}{2}\left(\partial_\mu A_\nu\, \partial^\mu A^\nu + a\partial_\mu A^\mu\, \partial_\nu A^\nu + bA_\mu A^\mu\right).$$

In $d+1$ dimensions there are $d+1$ Euler-Lagrange equations: one for each component of A^μ. They are,

$$\begin{aligned} 0 &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A^\nu} \\ &= \pm\left(\partial_\mu(\partial^\mu A_\nu + a\partial_\nu A^\mu) - bA_\nu\right). \end{aligned}$$

We look for plane wave solutions times a **polarization vector** ε^μ independent of x:

$$A^\mu(x) = \varepsilon^\mu(k)\, e^{-ik\cdot x},$$

where $k \cdot x = -\omega t + \mathbf{k}\cdot\mathbf{x}$. We are using notation where x and k represent $(d+1)$-component vectors, and we reserve the notation $\mathbf{x}$ and $\mathbf{k}$ for the spatial components.

Plugging the plane wave *ansatz* into the equations of motion, we obtain:

$$0 = (-ik_\mu)\cdot\left(-ik^\mu\,\varepsilon_\nu\, e^{-ik\cdot x} + a(-ik_\nu)\,\varepsilon^\mu\, e^{-ik\cdot x}\right) - b\,\varepsilon_\nu\, e^{-ik\cdot x}$$

$$
\begin{aligned}
&= e^{-ik\cdot x}\left(-k^2\varepsilon_\nu - a\,(\varepsilon\cdot k)\,k_\nu - b\,\varepsilon_\nu\right). \\
&= e^{-ik\cdot x}\left((\omega^2-\mathbf{k}^2)\varepsilon_\nu - a\,\varepsilon\cdot k\,k_\nu - \ \varepsilon_\nu\right).
\end{aligned}
$$

We can distinguish two different kinds of modes:

- **Longitudinal modes**: $\varepsilon_\nu \propto k_\nu$
- **Transverse modes**: $\varepsilon \cdot k = 0$

These conditions are Lorentz invariant. We will see that for $b \neq 0$ in the Lagrangian the modes are massive, so that it makes sense to speak about the rest frame of the wave. In the rest frame, the longitudinal condition is $\varepsilon \propto (\omega, \mathbf{0})$. This is a rotational scalar, and the longitudinal mode can be thought of as a scalar field.

The transverse modes in the rest frame satisfy $\varepsilon_0\,\omega = 0$, so only the spatial components ε are nonvanishing. These d components form a vector under d-dimensional rotations, and will be the modes of interest for us as they are manifestly different than the scalar mode.

We will consider the two kinds of modes separately. First, the longitudinal mode: The equations of motion with $\varepsilon^\nu \propto k^\nu$ are:

$$(\omega^2 - \mathbf{k}^2)\,k_\nu + a\,(\omega^2 - \mathbf{k}^2)\,k_\nu - b\,k_\nu = 0,$$

so that,

$$(\omega^2 - \mathbf{k}^2) = \frac{b}{1+a} \equiv m_L^2.$$

Hence, the longitudinal mode has the dispersion relation for a field of mass m_L. It is equivalent to a scalar field with the same mass.

Next, the transverse mode: Setting $\varepsilon \cdot k = 0$ in the equations of motion,

$$(\omega^2 - \mathbf{k}^2)\,\varepsilon_\nu - b\,\varepsilon_\nu = 0,$$

so that,

$$\omega^2 - \mathbf{k}^2 = b \equiv m_T^2.$$

Hence, the transverse modes have the dispersion relation for waves of mass m_T. Notice that by the equations of motion for the transverse mode,

$$\partial_\mu \partial^\mu A_\nu = m_T^2 A_\nu,$$

each component of the transverse vector field satisfies the Klein-Gordon equation for a field of mass m_T.

Decoupling of the longitudinal mode

The longitudinal mode is a scalar which we would prefer to separate from the rest of our discussion of the vector field. We can arrange for there to be no nonvanishing longitudinal solution if we choose $a = -1$ in the Lagrangian. Roughly speaking, the longitudinal mass $m_L \to \infty$ so that the longitudinal mode becomes inaccessible. Then we redefine the transverse mass to be m, so that $b = m^2$.

The Lagrangian for the transverse vector field is then,

$$\mathcal{L} = \pm\frac{1}{2}\left(\partial_\mu A_\nu\, \partial^\mu A^\nu - \partial_\mu A^\mu\, \partial_\nu A^\nu + m^2\, A_\mu A^\mu\right).$$

We can simplify the form of the Lagrangian by defining the **field strength** tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The field strength is antisymmetric under exchange of its indices. Up to addition of a total derivative, the Lagrangian can be written as,

$$\mathcal{L} = \pm\left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2}\, A_\mu A^\mu\right).$$

This is the **Proca Lagrangian**. It describes a massive vector field. In terms of the field strength, the equations of motion for the massive vector field are,

$$\partial_\mu F^{\mu\nu} = m^2\, A^\nu.$$

In terms of A^μ, the following is an identity, known as the **Bianchi identity**:

$$\epsilon^{\mu_1 \mu_2 \cdots \mu_{d+1}}\, \partial_{\mu_1} F_{\mu_2 \mu_3} = 0.$$

This follows from the antisymmetry of $\epsilon^{\mu_1 \cdots \mu_{d+1}}$ and the symmetry of the mixed partial derivatives. While it is an identity in terms of A^ν, in terms of the field strength it appears as a constraint.

Recall that the field due to a source falls off exponentially with distance from the source for a massive field. Electromagnetism is a long range force, so we take the mass $m \to 0$ to obtain a theory that could describe electromagnetism. This is, in fact, the Lorentz-covariant form of Maxwell's theory.

Specifying initial conditions

Expanding the Lagrangian density into terms involving time components and those only involving spatial components,

$$\mathcal{L} = \pm\frac{1}{2}\left(F_{0i}F^{0i} + \frac{1}{2}F_{ij}F^{ij} + m^2\, A_i A^i + m^2\, A_0 A^0\right).$$

The $(0i)$ component of the field strength is $F_{0i} = \partial_0 A_i - \partial_i A_0$. The **canonical momentum** conjugate to the field A^i is:

$$\Pi^i \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = \pm F^{0i}.$$

But, the canonical momentum conjugate to A^0 is,

$$\Pi^0 \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0\ !$$

The Hamiltonian is expressed in terms of the canonical momenta, so for consistency we should check that we do not need to know A^0 or $\partial_0 A^0$ in order to completely specify the dynamics. More precisely, a complete set of initial conditions should be specified by A^i and Π^i in order for the Hamiltonian to consistently describe the dynamics.

Luckily, this is the case. Maxwell's equations are a set of linear second order partial differential equations, so certainly the collection of $A^0, \partial_0 A^0, A^i, \partial_0 A^i$ supply a complete set of initial value data.

Recall that the equations of motion are:

$$\partial_\mu F^{\mu\nu} = m^2\, A^\nu.$$

Taking another derivative, we have,

$$\partial_\nu \partial_\mu F^{\mu\nu} = m^2\, \partial_\nu A^\nu.$$

But the left hand side vanishes because $F^{\mu\nu}$ is antisymmetric under $\mu \leftrightarrow \nu$ and the mixed partial derivatives are symmetric.

So we conclude that for the massive vector field,

$$\partial_\nu A^\nu = 0.$$

This is often called the Lorentz gauge condition. It is a necessary condition for the free massive vector field, but is not required in massless electrodynamics.

Notice also that with the Lorentz condition the equations of motion are,

$$\begin{aligned} m^2 A^\nu &= \partial_\mu F^{\mu\nu} \\ &= \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\ &= \partial_\mu \partial^\mu\, A^\nu \end{aligned}$$

Hence, each component of A^ν satisfies the Klein-Gordon equation for a field of mass m.

From the Lorentz gauge condition,

$$\partial_\mu A^\mu = 0 \rightarrow \partial_0 A^0 = -\partial_i A^i.$$

Hence, knowledge of $A^i(\mathbf{x})$ at a fixed time determines $\partial_0 A^0$ at that time.

From the time component of Maxwell's equations,

$$\partial_i F^{i0} = m^2 A^0 \rightarrow A^0 = \frac{1}{m^2}\, \partial_i F^{i0}.$$

Hence, knowledge of $\Pi^i = \mp F^{i0}$ at an initial time determines A^0 at that time.

From the definition of the field strength,

$$F_{0i} = \partial_0 A_i - \partial_i A_0 \to \partial_0\, A_i = F_{0i} + \partial_i A_0.$$

Hence, knowledge of $\Pi_i(\mathbf{x})$ and $A_0(\mathbf{x})$ at an initial time determine $\partial_0 A_i$ at that time.

So we have shown that, thankfully, A^i and Π_i provide a complete set of initial value data for solutions to the theory. This means it is okay that Π^0 vanishes. A^0 is not an independent field, and is specified once we know A^i. The vanishing of Π^0 is directly related to the fact that we have eliminated the longitudinal mode from the theory.

The Hamiltonian for the vector field

The Hamiltonian density is given by,

$$\begin{aligned}
\mathcal{H} &= \sum_i \Pi^i\, \partial_0 A_i - \mathcal{L} \\
&= \pm F^{0i}\, \partial_0 A_i - \mathcal{L} \\
&= \pm F^{0i} F_{0i} \pm F^{0i}\, \partial_i A_0 - \mathcal{L} \\
&= \pm F^{0i} F_{0i} \mp \left(\partial_i\, F^{0i}\right) A_0 - \mathcal{L} + \text{tot. deriv.}
\end{aligned}$$

In the last line we anticipated integrating by parts by adding a total derivative that is irrelevant when we integrate over $d^d x$ to obtain the Hamiltonian. Now recall that from the equations of motion,

$$A^0 = -\frac{1}{m^2}\, \partial_i\, F^{0i}.$$

Hence,

$$\begin{aligned}
\mathcal{H} &= \pm F^{0i} F_{0i} \pm m^2\, A_0\, A^0 - \mathcal{L} \\
&= \pm \frac{1}{2} \left(F_{0i} F^{0i} - \frac{1}{2} F_{ij} F^{ij} - m^2\, A_i A^i + m^2\, A_0 A^0 \right).
\end{aligned}$$

Considering that the indices are lowered with the Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \ldots)$, you can check that each term in the parentheses is negative. Hence, the appropriate choice of overall sign in order for the Hamiltonian to be bounded below is the negative sign.

So, we have now completely determined the Lagrangian for massive electrodynamics:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu .$$

Number of degrees of freedom

In $d+1$ dimensions, how many propagating degrees of freedom are there? Equivalently, how many independent plane wave solutions are there for a given wavevector k?

Well, the plane wave solutions were of the form, $A^\nu = \varepsilon^\nu e^{-ik\cdot x}$. The solutions are transverse, so there is one condition $\varepsilon \cdot k = 0$. Hence, there are $(d+1) - 1 = d$ independent polarizations of the massive vector field. The missing polarization is, of course, the longitudinal polarization which we have eliminated from the theory by our choice of the Lagrangian.

Electromagnetism as a vector field

The electromagnetic field propagates at long distances. (That is why your hair is attracted to a rubber balloon.) As a result, the mass of the electromagnetic field must be very small. Recall that each component of the massive vector field satisfies the Klein-Gordon equation, so that the field falls off exponentially with distance unless $m \to 0$.

Incidentally, this is why the weak interactions are a short ranged interaction. The carriers of the interaction are the W and Z bosons, which are vector fields with masses around 80 GeV/c^2 and 91 GeV/c^2, respectively. Their masses set the distance scale beyond which the weak interactions are unimportant.

So, in Maxwell's theory of electrodynamics $m = 0$. Coupling to a source is analogous to how we coupled the scalar field to a source. The difference is that now the source is a vector with $d+1$ components, J^ν, and is called the **current**. We identify the time-component of the current with the usual electric charge density ρ, and the spatial components with the spatial electric current $\mathbf{J}$, so that $J^\nu \sim (\rho, \mathbf{J})$.

The current appears in Maxwell's equations in the following way:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -4\pi J^\nu \\ \epsilon^{\mu_1\cdots\mu_{d+1}} \partial_{\mu_1} F_{\mu_2\mu_3} &= 0. \end{aligned}$$

The source appears only in the first of the equations because the second is a constraint equation, and does not reflect the dynamics. Maxwell's equations in the presence of a source follow from the Lagrangian density,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 4\pi J_\nu A^\nu .$$

To relate the covariant form of Maxwell's equations to the version of Maxwell's equations we first learn about in physics courses, we identify the components of the field strength $F^{\mu\nu}$ in 3+1 dimensions with the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. The identifications is,

$$\begin{aligned} E_i &= F_{i0} \\ B_i &= \frac{1}{2}\epsilon_{ijk} F^{jk} . \end{aligned}$$

This identification only makes sense in $3+1$ dimensions, and we will generalize it in a little while. Recall that the ϵ symbol is the totally antisymmetric tensor, which in this case is invariant under rotations. It is normalized by $\epsilon^{12\cdots d} = +1$. Lowering the indices with the Euclidean metric δ_{ij}, we also have $\epsilon_{12\cdots d} = +1$.

In order to invert the definition of B^i, it is useful to derive some identities involving products of the ϵ symbol.

First, contracting all indices of the product of two ϵ's gives the trace:

$$\epsilon_{i_1\cdots i_d}\, \epsilon^{i_1\cdots i_d} = d!\, \epsilon_{12\cdots d}\, \epsilon^{12\cdots d} = d!\ .$$

There are $d!$ terms each contributing 1 to the trace, corresponding to the $d!$ ways of permuting the indices $i_1 \cdots i_d$.

Contracting all but one index on each ϵ,

$$\epsilon_{i_1\cdots i_{d-1} j}\, \epsilon^{i_1\cdots i_{d-1} k} = (d-1)!\, \delta_j^k\ .$$

This structure can be easily verified: If $j = k$ then there are $(d-1)!$ terms that contribute equally to the sum over the contracted indices, corresponding to the $(d-1)!$ permutations of the indices $i_1 \cdots i_{d-1}$. If $j \neq k$ then the sum vanishes because each term in the sum involves at least one ϵ with repeated indices, which vanishes by the antisymmetry of ϵ. (For example, switching the first two indices of $\epsilon^{11 i_3 \cdots i_d}$, we get $\epsilon^{11 i_3 \cdots i_d} = -\epsilon^{11 i_3 \cdots i_d}$, so that $\epsilon^{11 i_3 \cdots i_d} = 0$.)

Similarly, by examining the symmetry structure of the product of ϵ's with two free indices on each ϵ, you can convince yourself that:

$$\epsilon_{i_1 \cdots i_{d-2} jk} \epsilon^{i_1 \cdots i_{d-2} lm} = \# \left(\delta^l_j \, \delta^m_k - \delta^l_k \, \delta^m_j \right).$$

You can check that the right hand side has the correct symmetry properties under exchange of indices. To evaluate the coefficient, we take the trace:

$$d! = \epsilon_{i_1 \cdots i_d} \, \epsilon^{i_1 \cdots i_d} = d! \, \epsilon_{12 \cdots d} \, \epsilon^{12 \cdots d} = \# \left(d^2 - d \right).$$

Hence, $\# = d!/(d^2 - d) = (d-2)!$. So,

$$\epsilon_{i_1 \cdots i_{d-2} jk} \epsilon^{i_1 \cdots i_{d-2} lm} = (d-2)! \, \left(\delta^l_j \, \delta^m_k - \delta^l_k \, \delta^m_j \right).$$

In three spatial dimensions, the relevant relation is,

$$\epsilon_{ijk} \epsilon^{ilm} = \left(\delta^l_j \, \delta^m_k - \delta^l_k \, \delta^m_j \right).$$

Now, multiplying the definition of B_i by ϵ^{ilm} and summing over i, we obtain,

$$\begin{aligned}
\epsilon^{ilm} \, B_i &= \frac{1}{2} \, \epsilon^{ilm} \, \epsilon_{ijk} \, F^{jk} \\
&= \frac{1}{2} \left(\delta^l_j \, \delta^m_k - \delta^l_k \, \delta^m_j \right) \, F^{jk} \\
&= \frac{1}{2} \left(F^{lm} - F^{ml} \right) \\
&= F^{lm}.
\end{aligned}$$

Putting it all together, the components of the field strength tensor in $3+1$ dimensions has the form,

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

With this definition, Maxwell's equations take the usual form: $\partial_\mu F^{\mu\nu} = -4\pi J^\nu$ implies the two equations,

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J} \end{aligned}$$

The Bianchi identity $\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu F_{\rho\sigma} = 0$, implies the remaining two of Maxwell's equations,

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned}$$

In more than $3+1$ dimensions, we can still define the electric field as $E_i = F_{i0}$, but the magnetic field becomes,

$$B_{i_1 \cdots i_{d-2}} = \frac{1}{2(d-2)!}\epsilon_{i_1 \cdots i_d} F^{i_{d-1} i_d}.$$

Hence, the magnetic field in d spatial dimensions is an antisymmetric tensor with $d-2$ indices, also known as a $(d-2)$**-form**.

References

[1] http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Maxwell.html
A short bio of James Clerk Maxwell.

[2] J.D. Jackson, *Electrodynamics*