Physics 690/482-02 Spring, 2005 Josh Erlich

Notes: Scalar fields in compact extra dimensions

Solutions to the massive wave equation on a torus

Consider a spacetime with three large spatial dimensions and $d-3$ compact extra dimensions forming a $d-3$ dimensional untwisted torus T^{d-3}. The torus is a product of circles, $T^n = S^1 \times S^1 \times \cdots \times S^1$, and has a flat metric (because each of the independent circle directions is flat).

Recall the massive wave equation,

$$-\partial_t^2 \varphi + \nabla_d^2 \varphi - m^2 \varphi = 0.$$

The solutions are plane waves, which are the real and imaginary parts of $\varphi(x^\mu) = e^{-ik\cdot x}$. (We are using the notation $k \cdot x = k_\mu x^\mu = -\omega t + \mathbf{k}\cdot\mathbf{x}$.) We will distinguish between the three large spatial dimensions $\mathbf{x}$ and the $d-3$ compact dimensions x^i.

If the circles forming the torus have sizes R_i, $i = 1, \ldots, d-3$, then there are periodic boundary conditions $\varphi(\mathbf{x}, x^i) = \varphi(\mathbf{x}, x^i + 2\pi n^i R^i)$. This requires that the components of the wavenumber in the compact dimensions be quantized:

$$k^i = n^i / R^i,$$

for some integers n^i.

Plugging the plane wave solution into the wave equation yields the dispersion relation:

$$\begin{aligned} \omega^2 &= \mathbf{k}^2 + \sum_{i=1}^{d-3} (k^i)^2 + m^2. \\ &= \mathbf{k}^2 + \sum_i \left(\frac{n_i}{R_i}\right)^2 + m^2. \end{aligned}$$

The generic solution to the wave equation is a sum over the plane wave solutions:

$$\varphi(\mathbf{x}, x^i, t) = \sum_{\{n_i\}} \prod_{i=1}^{d-3} \left(\frac{1}{2\pi R_i}\right) \int \frac{d^{3+1}k}{(2\pi)^4}\, a_{\{n_i\}}(\mathbf{k}, \omega) \exp\left[i\left(\omega t - \mathbf{k}\cdot\mathbf{x} - \sum_j \frac{n_j x^j}{R_j}\right)\right].$$

You will recognize this as a Fourier transform, where the infinitely large dimensions correspond to the continuous frequency ω and wavenumbers $\mathbf{k}$, and the compact dimensions correspond to the discrete wavenumbers $k_i = n_i/R_i$. The sum is over all possible choices of the $n_i = -\infty, \ldots, \infty$:

$$\sum_{\{n_i\}} = \sum_{i=1}^{d-3} \sum_{n_i=-\infty}^{\infty} .$$

The statement that φ is a real scalar field translates into the requirement that the Fourier coefficients satisfy $a_{\{n_i\}}(\mathbf{k}, \omega) = a_{\{-n_i\}}(-\mathbf{k}, -\omega)^*$, as can be seen by taking the complex conjugate of φ:

$$\begin{aligned}
\varphi(\mathbf{x}, x^i, t)^* &= \sum_{\{n_i\}} \prod_{i=1}^{d-3} \left(\frac{1}{2\pi R_i}\right) \int \frac{d^{3+1}k}{(2\pi)^4} a_{\{n_i\}}(\mathbf{k}, \omega)^* \exp\left[-i\left(\omega t - \mathbf{k}\cdot\mathbf{x} - \sum_j \frac{n_j x^j}{R_j}\right)\right] \\
&= \sum_{\{n_i\}} \prod_{i=1}^{d-3} \left(\frac{1}{2\pi R_i}\right) \int \frac{d^{3+1}k}{(2\pi)^4} a_{\{-n_i\}}(-\mathbf{k}, -\omega)^* \exp\left[+i\left(\omega t - \mathbf{k}\cdot\mathbf{x} - \sum_j \frac{n_j x^j}{R_j}\right)\right] .
\end{aligned}$$

Setting $\varphi = \varphi^*$, the desired relation follows.

Solutions in the presence of a source

In the presence of a static source, the equation of motion is:

$$(-\partial_t^2 + \nabla^2 + \sum_{i=1}^{d-3} \partial_i^2)\, \varphi - m^2 \varphi = \rho(\mathbf{x}, x^i),$$

where ∇ is now the gradient in the three noncompact dimensions.

Now we just repeat the steps we performed in the case of $d+1$ noncompact dimensions, with the following changes:

$$\begin{aligned}
\mathbf{k} &\rightarrow \{\mathbf{k}, k^i\} = \left\{\mathbf{k}, \frac{n_i}{R_i}\right\} \\
\int \frac{d^d k}{(2\pi)^d} &\rightarrow \int \frac{d^3 k}{(2\pi)^3} \sum_{\{n_i\}} \prod_{i=1}^{d-3} \left(\frac{1}{2\pi R_i}\right)
\end{aligned}$$

The factors of $\frac{1}{2\pi R_i}$ are the factors appearing in the inverse Fourier transform, and follow from the completeness relations for the discrete plane wave solutions (just as the factors of $\frac{1}{2\pi}$ followed from the completeness relations for the continuous plane

wave solutions). The relevant relations, given without proof, are:

$$\begin{aligned}\int_0^{2\pi R} dx\, \exp\left[i\frac{(n-m)\,x}{R}\right] &= 2\pi R\,\delta_{nm}\\ \sum_{n=-\infty}^{\infty} \exp\left[-i\frac{n(x-x')}{R}\right] &= 2\pi\,\delta(x-x')\end{aligned}$$

As a consequence of the completeness relations above, a consistent definition of the Fourier transform and its inverse for any function periodic in $x \to x + 2\pi R$ are:

$$\begin{aligned}f(x) &= \sum_{n=-\infty}^{\infty} \frac{e^{-inx}}{2\pi R}\,\tilde{f}_n\\ \tilde{f}_n &= \int dx\, e^{inx}\, f(x).\end{aligned}$$

Fourier transforming the equations of motion, we are led to the Fourier transformed solution,

$$\tilde{\varphi}(\mathbf{k}, \frac{n_i}{R_i}) = -\frac{\tilde{\rho}\left(\mathbf{k}, \frac{n_i}{R_i}\right)}{\mathbf{k}^2 + \sum_{i=1}^{d-3}\left(\frac{n_i}{R_i}\right)^2 + m^2}.$$

Taking the inverse Fourier transform, we then obtain,

$$\varphi(\mathbf{x}, x^i) = \sum_{\{n_i\}} \prod_{i=1}^{d-3} \left(\frac{\exp\left[-i\frac{n_i x^i}{R_i}\right]}{2\pi R_i}\right) \int \frac{d^3k}{(2\pi)^3} \frac{-\tilde{\rho}\left(\mathbf{k}, \frac{n_i}{R_i}\right) \exp\left[-i\mathbf{k}\cdot\mathbf{x}\right]}{\mathbf{k}^2 + \sum_{j=1}^{d-3}\left(\frac{n_j}{R_j}\right)^2 + m^2}.$$

Finally, replacing $\tilde{\rho}$ by its inverse Fourier transform, we obtain the desired result, which is simply the replacement of $\mathbf{k}$ by $\{\mathbf{k}, n_i/R_i\}$ in the solution we found with all noncompact dimensions:

$$\varphi(\mathbf{x}, x^i) = -\int d^d x'\, \rho(\mathbf{x}', x'^i) \sum_{\{n_i\}} \prod_{i=1}^{d-3} \left(\frac{\exp\left[-i\frac{n_i(x^i - x'^i)}{R^i}\right]}{2\pi R_i}\right) \int \frac{d^3k}{(2\pi)^3} \frac{\exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]}{\mathbf{k}^2 + \sum_{j=1}^{d-3}\left(\frac{n_j}{R_j}\right)^2 + m^2}.$$

Potential energy

The Hamiltonian is the same as we derived for the noncompact case:

$$H = \frac{1}{2}\int d^d x\, \rho(\mathbf{x}, x^i)\varphi(\mathbf{x}, x^i).$$

Inserting the solution we have found for φ in the presence of the source, we obtain:

$$\begin{aligned} H &= -\frac{1}{2}\int d^dx \int d^dx'\, \rho(\mathbf{x}, x^i)\rho(\mathbf{x}, x'^i) \sum_{\{n_i\}} \prod_{i=1}^{d-3} \left(\frac{\exp\left[-i\frac{n_i(x^i - x'^i)}{R^i}\right]}{2\pi R_i} \right) \\ &\quad \times \int \frac{d^3k}{(2\pi)^3} \frac{\exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]}{\mathbf{k}^2 + \sum_{j=1}^{d-3}\left(\frac{n_j}{R_j}\right)^2 + m^2}. \end{aligned}$$

From the Hamiltonian we read off the potential,

$$V(\mathbf{x}-\mathbf{x}') = -\sum_{\{n_i\}} \prod_{i=1}^{d-3} \left(\frac{\exp\left[-i\frac{n_i(x^i - x'^i)}{R^i}\right]}{2\pi R_i} \right) \int \frac{d^3k}{(2\pi)^3} \frac{\exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]}{\mathbf{k}^2 + \sum_{i=j}^{d-3}\left(\frac{n_j}{R_j}\right)^2 + m^2}$$

We recognize the integral as the Yukawa potential in three spatial dimensions. So we can write the d-dimensional potential as a sum of Yukawa potentials:

$$\begin{aligned} V(\mathbf{x}-\mathbf{x}') &= -\sum_{\{n_i\}} \prod_{i=1}^{d-3} \frac{\exp\left[-i\frac{n_i(x^i - x'^i)}{R^i}\right]}{2\pi R_i} \int \frac{d^3k}{(2\pi)^3} \frac{\exp\left[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right]}{\mathbf{k}^2 + \sum_{j=1}^{d-3}\left(\frac{n_j}{R_j}\right)^2 + m^2} \\ &= -\sum_{\{n_i\}} \prod_{i=1}^{d-3} \frac{\exp\left[-i\frac{n_i(x^i - x'^i)}{R^i}\right]}{2\pi R_i} \frac{e^{-m_{\{n_i\}}|x-x'|}}{4\pi|x-x'|}, \end{aligned}$$

where $m_{\{n_i\}}$ is the effective (3+1)-dimensional mass of the corresponding Kaluza-Klein mode,

$$m_{\{n_i\}} \equiv \left(\sum_{i=1}^{d-3} \left(\frac{n_i}{R_i}\right)^2 + m^2 \right)^{1/2},$$

and,

$$|x-x'| = \left((\mathbf{x}-\mathbf{x}')^2 + \sum_{i=1}^{d-3} (x^i - x'^i)^2 \right)^{1/2}.$$

Each Kaluza-Klein mode contributes a term to the potential, so it is as if there were an infinite number of fields with masses $m_{\{n_i\}}$, each giving rise to a potential between sources. From the 3+1 dimensional perspective, there is a tower of massive fields, and *the existence of that tower is tantamount to the existence of an extra dimension.*

The potential for large separations

For very large separations ($|\mathbf{x}-\mathbf{x}'| \gg R_i$ for all R_i), only the lowest mass Kaluza-Klein mode contributes because of the decaying exponential in the Yukawa potential. Hence, for large separations,

$$V(\mathbf{x}-\mathbf{x}') \simeq -\prod_{i=1}^{d-3}\left(\frac{1}{2\pi R_i}\right)\frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}.$$

The volume factor comes from the normalization of φ in different dimensions, as can be seen by examining the action. We won't worry about this right now. The rest is simply the Yukawa potential in 3 spatial dimensions for a field of mass m. Hence, we have derived the intuitive result that *by measuring the potential at separations much larger than the size of the compact extra dimensions, one could not determine that the extra dimensions exist.*

The potential for small separations

In the opposite limit, $\left((\mathbf{x}-\mathbf{x}')^2+\sum_i(x^i-x'i)^2\right)^{1/2} \ll R_i$, the sum over discrete wavenumbers $\{n_i\}$ becomes an integral:

$$\begin{aligned}\frac{n_i}{R_i} &\rightarrow k^i \\ \sum_{\{ni\}}\prod_{i=1}^{d-3}\left(\frac{1}{2\pi R_i}\right) &\rightarrow \int\frac{d^{d-3}k}{(2\pi)^{d-3}}\end{aligned}$$

With this substitution, the potential simply becomes that of a scalar field of mass m in $d+1$ noncompact dimensions. Hence, we cannot tell whether a dimension is compact or not by measuring the potential at small distances. If our universe closed closed back on itself after a very large distance, there's no way we would know it just by doing laboratory experiments.

In fact, some people have recently speculated that the universe does in fact close back on itself in a certain way. Such a topology to the universe could explain why the observed power of the cosmic microwave background at very large distances is lower than had been predicted.

Massless scalar potential and the search for extra dimensions

In the limit $m \to 0$, the potential varies as $V(\mathbf{x}-\mathbf{x}') \propto 1/|\mathbf{x}-\mathbf{x}'|^{d-2}$ in d noncompact spatial dimensions. There are several ways to derive this.

The behavior of the potential follows immediately from the expression of the Yukawa coupling in d-dimensions as an integral over k, because with $m = 0$ the only dimensionful quantity is $|\mathbf{x}-\mathbf{x}'|$, and counting factors of k (which have dimension 1/Length), the dependence on $|\mathbf{x}-\mathbf{x}'|$ follows.

Alternatively, one can use Gauss' law in d-dimensions:

$$\int (\nabla V) \cdot dS = \int d^d x \, \rho(x).$$

Performing the surface integral over a $d-1$-dimensional sphere gives,

$$(\partial_r V)(r^{d-1}\, \Omega_{d-1}) = \int d^d x \, \rho(x),$$

where Ω_{d-1} is the area of the $d-1$ dimensional unit sphere. Integrating with respect to r again gives the desired behavior of the potential.

We have seen that it is difficult to detect very small extra dimensions. But the behavior of the potential as a function of the number of dimensions provides a way to search for extra dimensions: Measure the force between two sources as a function of distance at smaller and smaller distances. For distances small enough that they are comparable to the size of the extra dimensions the higher mass Kaluza-Klein modes will contribute to the potential, and one would see the behavior of the force with distance change from a three dimensional to a higher dimensional behavior.

We will discuss other ways to experimentally search for extra dimensions later in the course.

The effective low energy theory

We expect that the massive Kaluza-Klein modes are not important at long distances, or equivalently at low energies. Hence, it makes sense to write down an action that includes only the lowest energy mode, with all of the $n_i = 0$.

Recall that we can always decompose a solution to the equations of motion as a sum over Kaluza-Klein modes. This is just the generic form of the solution we

wrote down before:

$$\varphi(\mathbf{x}, x^i, t) = \sum_{\{n_i\}} \prod_{i=1}^{d-3} \left(\frac{1}{2\pi R_i}\right) \int \frac{d^{3+1}k}{(2\pi)^4} a_{\{n_i\}}(\mathbf{k}, \omega) \exp\left[i\left(\omega t - \mathbf{k}\cdot\mathbf{x} - \sum_j \frac{n_j x^j}{R_j}\right)\right].$$

If we set the coefficients of all the Kaluza-Klein modes to zero except for the mode with $\{n_i\} = 0$, then we have the solution,

$$\begin{aligned} \varphi(\mathbf{x}, x^i, t) &= \prod_{i=1}^{d-3} \left(\frac{1}{2\pi R_i}\right) \int \frac{d^{3+1}k}{(2\pi)^4} a_{\{n_i=0\}}(\mathbf{k}, \omega) \exp\left[i\left(\omega t - \mathbf{k}\cdot\mathbf{x}\right)\right] \\ &\equiv \phi(\mathbf{x}, t). \end{aligned}$$

The action for the theory is,

$$S = -\int d^{d+1}x \, \frac{1}{2}\left((\partial_\mu \varphi)^2 + m^2\varphi^2\right).$$

Plugging the low energy solution for φ into the Lagrangian, we obtain the **low energy effective action**,

$$S_{eff} = -\prod_{i=1}^{d-3} (2\pi R_i) \int d^{3+1}x \frac{1}{2}\left((\partial_\mu \phi)^2 + m^2\phi^2\right).$$

The volume factor is related to the normalization of ϕ. By redefining $\phi' = \Pi_{i=1}^{d-3}(2\pi R_i)^{1/2}\,\phi$ we can absorb that factor. What remains is an action for a scalar field with mass m in 3+1 dimensions. This is consistent with the form of the potential at large distances.

We will be considering more complicated situations later, but we will often be interested in the low energy effective theory. The general procedure for constructing the low energy theory is this:

- Expand the fields as sums over Kaluza-Klein modes.
- Set the coefficients of all but the lowest mass modes to zero.
- Insert the result into the $d+1$ dimensional action.
- Integrate over the extra dimensions.
- Rescale the fields by a volume factor so that they have the canonical derivative terms in the resulting $3+1$ dimensional action.

Appendix: Area of the $(d-1)$-dimensional unit sphere

Recall that by Gauss' law we showed that,

$$\partial_r V(r) = \frac{\Omega_{d-1}}{r^{d-1}} \int d^d x \, \rho(x).$$

Ω_{d-1} is the area of the $(d-1)$-dimensional unit sphere. The easiest way to derive it is as follows:

Consider the Gaussian, $I \equiv \int_{-\infty}^{\infty} dx \, e^{-x^2}$. To evaluate the Gaussian we can square it and change coordinates to polar coordinates. Then,

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, e^{-(x^2+y^2)} \\ &= \int_0^{2\pi} d\theta \int_0^{\infty} dr \, r \, e^{-r^2} \\ &= \pi. \end{aligned}$$

Hence, $I = \sqrt{\pi}$. Next consider I^d for a positive integer d:

$$\begin{aligned} I^d = \pi^{d/2} &= \int d\Omega_{d-1} \int_0^{\infty} dr \, r^{d-1} \, e^{-r^2} \\ &= \Omega_{d-1} \frac{\Gamma(d/2)}{2} \end{aligned}$$

The Gamma function is defined as the integral,

$$\Gamma(x) \equiv 2 \int_0^{\infty} dr \, r^{2x-1} \, e^{-r^2}.$$

The area of the unit sphere is then,

$$\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

For integer or half integer x the Gamma function can be evaluated by integrating by parts repeatedly until there is either a single factor of r or no factors of r left in the integral. You can check that for integer values, $\Gamma(n) = (n-1)!$; and for half-integer values, $\Gamma\left(\frac{2n+1}{2}\right) = (2n-1)!!\sqrt{\pi}/2^n$.