Physics 690/482-02 Spring, 2005 Josh Erlich
Notes: Scalar fields in $d + 1$ dimensions

Lagrangian densities for scalar field theories

The action for any field theory must be invariant under the symmetries of the theory in order for the resulting equations of motion to have those symmetries. Recall that the volume element $d^{d+1}x\,\sqrt{|g|}$ is a scalar under general coordinate transformations. In flat spacetimes with metric $ds^2 = -dt^2 + \mathbf{dx}^2$, which we will consider first, the invariant volume element is simply $d^{d+1}x$. In order for the action to be invariant, the Lagrangian must be invariant up to addition of a total derivative. Hence, the Lagrangian should be a scalar.

Using the index notation we have developed, it is straightforward to construct scalars. Any tensor with no indices, or with all indices contracted, is a scalar.

Consider a scalar field $\varphi(x)$. The most general Lagrangian density $\mathcal{L}$ with no derivatives,

$$\mathcal{L}_{\text{no deriv}} = \sum_n a_n\,\varphi^n,$$

with constants a_n, is a scalar. The Lagrangian is the spatial integral of the Lagrangian density, $L = \int d^dx\,\mathcal{L}$; and the action is the time integral of the Lagrangian, $S = \int d^{d+1}x\,\mathcal{L}$.

Extremizing the action yields the Euler-Lagrange equations,

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = \frac{\partial\mathcal{L}}{\partial\varphi}.$$

The Euler-Lagrange equation for the Lagrangian $\mathcal{L}_{\text{noderiv}}$ are,

$$\sum_n n\,a_n\,\varphi^{n-1} = 0.$$

There are only static solutions, which are uninteresting from the perspective of physical systems.

Including derivatives, we must include at least two derivatives in order to form a Lorentz scalar. The simplest dynamical scalar field Lagrangian is,

$$L_{\text{two deriv}} = -\int d^d x\, \frac{1}{2}\partial_\mu\varphi\, \partial^\mu\varphi = -\int d^d x\, \frac{1}{2}\partial_\mu\varphi\, \partial_\nu\varphi\, \eta^{\mu\nu}.$$

The factor of 1/2 is arbitrary and was chosen for convenience. The minus sign was chosen so that the Hamiltonian is bounded below, as we will demonstrate soon.

The corresponding equations of motion are,

$$\partial_\mu\partial^\mu\, \varphi = (-\partial_t^2 + \nabla_d^2)\varphi = 0.$$

This is the **massless scalar wave equation**.

A basis of solutions of the wave equation are of the form $\varphi = A\,(e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} + c.c.)$ and $\varphi = iA\,(e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} - c.c.)$, where $c.c$ denotes the complex conjugate of the expression preceeding it. Plugging either of these ansatze into the wave equation yields the **dispersion relation** between the angular frequency ω and the d-dimensional wavenumber $\mathbf{k}$:

$$\omega^2 - \mathbf{k}^2 = 0.$$

The massless wave equation is easy to solve because it is a linear partial differential equation. We can add the term $-m^2\,\varphi^2/2$ to the Lagrangian density without changing the linearity of the equations of motion. This yields the Lagrangian for the **free massive scalar field**,

$$L = -\int d^d x\, \frac{1}{2}\left(\partial_\mu\varphi\, \partial^\mu\varphi + m^2\,\varphi^2\right).$$

The resulting equations of motion are the **massive scalar wave equation**,

$$(\partial_\mu\partial^\mu - m^2)\,\varphi = (-\partial_t^2 + \nabla_d^2 - m^2)\,\varphi = 0.$$

The massive wave equation has plane wave solutions which are the real and imaginary parts of $\varphi = A\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}$ as before, but with dispersion relation,

$$\omega^2 = \mathbf{k}^2 + m^2.$$

The massive wave equation is also known as the **Klein-Gordon equation**, and it arises as the relativistic version of Schrödinger's equation. Recall that the

Schrödinger equation for a massive particle with no interctions could be thought of as a wave equation for a wavefunction with plane wave solutions $\psi(\mathbf{x},t) = e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}$. The dispersion relation for the wavefunction follows from the de Broglie relations,

$$\begin{aligned} E &= \hbar\omega \\ \mathbf{p} &= \hbar\mathbf{k}, \end{aligned}$$

together with the Newtonian relation between energy and momentum,

$$E = \frac{\mathbf{p}^2}{2m}.$$

Putting it all together, consistency with the de Broglie relations require the dispersion relation,

$$\hbar\,\omega = \frac{\hbar^2\,\mathbf{k}^2}{2m}.$$

Taking derivatives of the plane wave solution, the correct dispersion relation follows from the Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\,\psi.$$

For a relativistic particle, $E^2 = \mathbf{p}^2 + m^2$, which, together with the de Broglie relations, yields the relativistic dispersion relation,

$$\hbar^2\omega^2 = \hbar^2\mathbf{k}^2 + m^2.$$

Taking derivatives of the plane wave, we find an equation for the wavefunction consistent with the dispersion relation:

$$\hbar^2\,(-\partial_t^2 + \nabla^2)\psi = m^2\psi.$$

This is the Klein-Gordon equation, and is the beginning of any discussion of relativistic quantum mechanics. Setting $\hbar = 1$, you will recognize the massive scalar wave equation.

Massive Scalar Green's function

Suppose the scalar field has a source $\rho(\mathbf{x})$, modifying the wave equation to,

$$(-\partial_t^2 + \nabla^2 - m^2)\varphi = \rho(\mathbf{x}).$$

Notice that for a static solution with $\varphi = \varphi(\mathbf{x})$, and with $m = 0$, this is the Poisson equation, $\nabla^2\varphi = \rho(\mathbf{x})$, for the gravitational or electric potential φ. Hence, a static solution to the massive scalar wave equation with a source is a generalization of the gravitational potential to the case of a massive gravitational field. (*Note:* The gravitational potential is a Newtonian concept, and we will generalize this discussion to GR later in the course.)

As we will see, the static solution is related to the energy contained in the field due to the source just as the gravitational potential is related to the energy in the gravitational field in the presence of a source. First let's find the solution to the sourced wave equation.

We are looking for static solutions, so $\partial_t^2\varphi = 0$. Fourier transforming the solution, we define $\tilde{\varphi}(\mathbf{k})$:

$$\tilde{\varphi}(\mathbf{k}) = \int d^dx\, e^{i\mathbf{k}\cdot\mathbf{x}}\,\varphi(\mathbf{x}).$$

Multiplying the scalar wave equation by $e^{i\mathbf{k}\cdot\mathbf{x}}$ and integrating over $\mathbf{x}$, we have:

$$\begin{aligned}\int d^dx\, e^{i\mathbf{k}\cdot\mathbf{x}}\,(\nabla^2 - m^2)\varphi(\mathbf{x}) &= \int d^dx\, e^{i\mathbf{k}\cdot\mathbf{x}}\,\rho(\mathbf{x}) \\ = \int d^dx\,(-\mathbf{k}^2 - m^2)e^{i\mathbf{k}\cdot\mathbf{x}}\,\varphi(\mathbf{x}) &\equiv \tilde{\rho}(\mathbf{k}) \\ = -(\mathbf{k}^2 + m^2)\,\tilde{\varphi}(\mathbf{k}).\end{aligned}$$

Hence,

$$\tilde{\varphi}(\mathbf{k}) = \frac{-\tilde{\rho}(\mathbf{k})}{\mathbf{k}^2 + m^2},$$

and the solution for $\varphi(\mathbf{x})$ is given by the inverse Fourier transform,

$$\begin{aligned}\varphi(\mathbf{x}) &= \int \frac{d^dk}{(2\pi)^d}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\tilde{\varphi}(\mathbf{k}) \\ &= -\int \frac{d^dk}{(2\pi)^d}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{\tilde{\rho}(\mathbf{k})}{\mathbf{k}^2 + m^2}.\end{aligned}$$

$$= -\int \frac{d^d k}{(2\pi)^d}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{1}{\mathbf{k}^2 + m^2} \int d^d x'\, e^{i\mathbf{k}\cdot\mathbf{x}'}\, \rho(\mathbf{x}').$$

Notice that $\phi(\mathbf{x})$ is guaranteed to be real by the reality of $\rho(\mathbf{x})$. In terms of the Fourier transformed field and source, the statement of reality is that $\tilde{\varphi}(\mathbf{k}) = \tilde{\varphi}(\mathbf{k})^*$, which follows from the similar statement for $\tilde{\rho}(\mathbf{k})$ and the form of our solution above.

We can now determine the **Green's function** $G(\mathbf{x}, \mathbf{x}')$ for the massive scalar wave equation, via the definition

$$\varphi(\mathbf{x}) = \int d^d x'\, G(\mathbf{x}, \mathbf{x}')\, \rho(\mathbf{x}').$$

Comparing with the solution for $\varphi(\mathbf{x})$ above, we find,

$$G(\mathbf{x}, \mathbf{x}') = -\int \frac{d^d k}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}}{\mathbf{k}^2 + m^2}.$$

The Green's function is the solution to the wave equation for a delta-function source, $\rho(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}')$, and once the Green's function is known the solution to the wave equation can be found for any source via the relations above.

The Yukawa Potential

It is of interest to know how the Hamiltonian varies as the source is varied. This information is contained in the **potential** $V(\mathbf{x} - \mathbf{x}')$.

Recall that the Hamiltonian in classical mechanics is given in terms of the Lagrangian by,

$$H = \sum_a p^a \dot{q}^a - L[q^a, p^a],$$

where q^a are the coordinates and $p^a = \partial L / \partial \dot{q}^a$ are the corresponding canonical momenta.

The correspondence between particle mechanics and field theory can be summarized as

$$\begin{cases} q^a(t) \to \phi(\mathbf{x}, t) \\ a \to \mathbf{x}. \\ t \to t \end{cases}$$

The field theory Hamiltonian is then,

$$H = \int d^d x \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \partial_t \phi - \mathcal{L} \right).$$

For the massive scalar field, the Hamiltonian is then,

$$H = \int d^d x \, \frac{1}{2} \left((\partial_t \varphi)^2 + (\nabla \varphi)^2 + m^2 \varphi^2 + 2\rho(\mathbf{x})\varphi \right).$$

Note that in the absence of the source, the Hamiltonian is positive definite. This is how the signs of the terms in the Lagrangian were chosen.

For our static solution, the time derivative vanishes, and integrating the spatial derivative term by parts yields,

$$H = \int d^d x \, \frac{1}{2} \varphi \left(\left(-\nabla^2 + m^2 \right) \varphi + 2\rho(\mathbf{x}) \right).$$

Note that in the absence of the source, the Hamiltonian is positive definite. This is how the signs of the terms in the Lagrangian were chosen.

Using the equation of motion, $\left(-\nabla^2 + m^2 \right) \varphi = -\rho(\mathbf{x})$, the Hamiltonian becomes,

$$H = \int d^d x \, \frac{1}{2} \rho \varphi.$$

We now insert our solution for φ:

$$\begin{aligned} H &= \frac{1}{2} \int d^d x \rho(\mathbf{x}) \left(-\int \frac{d^d k}{(2\pi)^d} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{1}{\mathbf{k}^2 + m^2} \int d^d x' \, e^{i\mathbf{k}\cdot\mathbf{x}'} \rho(\mathbf{x}') \right) \\ &= -\frac{1}{2} \int d^d x \, d^d x' \rho(\mathbf{x}) \, \rho(\mathbf{x}') \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\mathbf{k}^2 + m^2}. \end{aligned} \tag{1}$$

The potential energy $V(\mathbf{x}-\mathbf{x}')$ due to the presence of the source is obtained by noticing that the Hamiltonian is of the form,

$$H = \frac{1}{2}\int d^d x\, d^d x'\, \rho(x)\, \rho(x')\, V(\mathbf{x}-\mathbf{x}').$$

We will soon see that $V(\mathbf{x}-\mathbf{x}')$ is a function only of $|\mathbf{x}-\mathbf{x}'|$. For a source localized around two positions, say $\mathbf{x_1}$ and $\mathbf{x_2}$, the Hamiltonian has four contributions: two are self energies with contribution from $\rho(\mathbf{x_1})^2\, V(\mathbf{0})$ and $\rho(\mathbf{x_2})^2\, V(\mathbf{0})$; the other two are equal and contribute $\rho(\mathbf{x_1})\rho(\mathbf{x_2})V(\mathbf{x_1}-\mathbf{x_2})$ to the energy of the system. The latter term is the only one that varies as $\mathbf{x_1}$ and $\mathbf{x_2}$ are varied. Hence, with the factor of $1/2$ factored out of the Hamiltonian and the factor of 2 from the two equal contributions, $V(\mathbf{x_1}-\mathbf{x_2})$ has the usual interpretation of the scalar potential.

So we have found the potential for the massive scalar field in d-dimensions:

$$V(\mathbf{x}-\mathbf{x}') = -\int \frac{d^d k}{(2\pi)^d}\, \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\mathbf{k}^2+m^2}.$$

Notice that for the free field, $V(\mathbf{x}-\mathbf{x}') = G(\mathbf{x},\mathbf{x}')$ found earlier.

It remains for us to perform the d-dimensional integral over $\mathbf{k}$, which we will complete for the case of greatest interest, $d=3$.

We first write the integral in d-dimensional spherical coordinates, where we choose the angle θ to be the angle between $\mathbf{k}$ and $\mathbf{x}-\mathbf{x}'$.

We have,

$$V(\mathbf{x}-\mathbf{x}') = -\int_0^\infty dk\, d\Omega_{d-1}\, \frac{1}{(2\pi)^d}\, e^{-ik|\mathbf{x}-\mathbf{x}'|\cos\theta} \frac{1}{\mathbf{k}^2+m^2},$$

where $d\Omega_{d-1}$ is the volume element of the unit $d-1$ dimensional sphere.

Notice that the potential is a function of $|\mathbf{x}-\mathbf{x}'|$, as expected by rotational invariance. Notice also that as $m \to 0$, on dimensional grounds we must have

$$V(|\mathbf{x}-\mathbf{x}'|) \propto 1/|\mathbf{x}-\mathbf{x}'|^{d-2}.$$

This can also be seen via Gauss' law applied to the Poisson equation in d-dimensions.

The θ part of the integral will be of the form,

$$V(\mathbf{x}-\mathbf{x}') = -\int_0^\infty dk \int \frac{d\Omega_{d-1}}{d\theta} \int_0^\pi d\theta \, \frac{1}{(2\pi)^d} \, (\sin\theta)^{d-2} \, \frac{e^{-ik|\mathbf{x}-\mathbf{x}'|\cos\theta}}{\mathbf{k}^2+m^2},$$

At this point we will specify $d = 3$ and obtain an explicit form for the potential. So, substituting $d = 3$ into the potential, we have,

$$\begin{aligned} V(\mathbf{x}-\mathbf{x}') &= -\int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\phi \, \frac{1}{(2\pi)^3} \, k^2 \, \sin\theta \, \frac{e^{-ik|\mathbf{x}-\mathbf{x}'|\cos\theta}}{\mathbf{k}^2+m^2} \\ &= -\frac{2\pi}{(2\pi)^3} \int_0^\infty dk \, \frac{1}{k^2+m^2} \, \frac{(e^{-ik|\mathbf{x}-\mathbf{x}'|} - e^{ik|\mathbf{x}-\mathbf{x}'|})}{-ik|\mathbf{x}-\mathbf{x}'|} \\ &= \frac{1}{(2\pi)^2 \, |\mathbf{x}-\mathbf{x}'|} \int_{-\infty}^{\infty} dk \frac{k^2 \, e^{ik\,|\mathbf{x}-\mathbf{x}'|}}{-ik\,(k^2+m^2)} \end{aligned}$$

We are left with an integral which can be performed easily by analytically continuing the integrand into the complex k-plane. Notice that the exponential in the integrand vanishes when $k \to +i\infty$, so that we can close the contour with an arc at infinity in the upper half of the complex plane.

We then use the residue theorem:

$$\oint dk \, \frac{f(k)}{k-k_0} = 2\pi i \, f(k_0),$$

for the integral around a simple pole. Our integral encircles a pole at im, so we obtain:

$$\begin{aligned} V(\mathbf{x}-\mathbf{x}') &= \frac{1}{(2\pi)^2 \, |\mathbf{x}-\mathbf{x}'|} \int_{-\infty}^{\infty} dk \frac{k^2 \, e^{ik\,|\mathbf{x}-\mathbf{x}'|}}{-ik\,(k+im)(k-im)} \\ &= \frac{1}{(2\pi)^2 \, |\mathbf{x}-\mathbf{x}'|} \, \frac{2\pi i \, (im)^2 \, e^{i(im)\,|\mathbf{x}-\mathbf{x}'|}}{-i(im)(2im)} \\ &= -\frac{1}{4\pi} \frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \end{aligned}$$

This is the Yukawa potential. Notice that it is real, so by its equivalence with the Green's function $G(\mathbf{x}, \mathbf{x}')$ as discussed earlier, the solution we derived for the scalar field in the presence of the source is also real. In other words, it was okay for us to have been using the complex plane waves as a basis of solutions: we automatically obtained a real solution in the presence of a real source.

The Yukawa potential is the potential carried by a massive scalar field in the presence of a source. Notice that as $m \to 0$ we recover the Coulomb potential, except *with the opposite sign.* This is the well-known fact that gravity is an attractive force, while in electromagnetism like charges repel.

In the presence of compact extra dimensions, each Kaluza-Klein mode contributes to the potential a term proportional to the Yukawa potential for a field with the corresponding Kaluza-Klein mass. That's what's next...