Physics 690/482-02 Spring, 2005 Josh Erlich

Notes: Geometry II

Euclidean geometries are flat. This means there is a coordinate system (x^1, x^2) in which the metric is the identity matrix δ_{ij}. In a crazy coordinate system it may not be obvious that the metric describes a flat space, so it would be nice to have a criterion that distinguishes flat from non-flat spaces. That criterion will be vanishing or non-vanishing of the **curvature tensor**.

Examples of flat spaces:
One dimension: Line, Circle
Two dimensions: The plane, cone (except at the singularity), cylinder, torus

Examples of non-flat spaces:
One dimension: None without boundaries or singularities
Two dimensions: Sphere

Metric on the sphere of radius R: $ds^2 = R^2\, d\theta^2 + R^2 \sin^2\theta\, d\phi^2$
There is no coordinate system in which $ds^2 = (dx^1)^2 + (dx^2)^2$.

Note that the metric on the 2-sphere is the same as the metric of flat three-dimensional space in spherical coordinates with constant $r = R$. The metric on the 2-sphere is said to be **induced** by the Euclidean metric. It is important to realize that we did not have to think of the embedding of the sphere in the third dimension to describe its geometry. The two-dimensional metric on the sphere provides everything we need to know about the geometry, at least locally.

Geodesics

Suppose we want to find the shortest or longest path between two points in a certain geometry. (In spacetime the path of shortest distance is taken by freely falling observers.) There are several ways to proceed. One way is to write $s = \int ds[x^i(\tau)]$ and use the calculus of variations to minimize s over the possible paths $x^i(\tau)$. Alternatively we can use the fact that (i) we know the answer already in a flat geometry, and (ii) there is a neighborhood of every point in a manifold in which the geometry looks locally flat. The consequence of fact (ii) is that locally there is a coordinate system in which the metric is Euclidean: $ds^2 = \overline{x}^i \overline{x}^j \, \delta_{ij}$. In Cartesian coordinates a straight line satisfies the equation, $\ddot{\overline{x}}^i = 0$. Then we can transform to a different coordinate system $\{x^i\}$ and thereby derive the equation describing geodesics in arbitrary coordinates. We will instead proceed with the calculus of variations approach because the techniques used are similar to those of Lagrangian field theories which we will need later in the course. We want to find the path which makes the path length stationary under a change in path $\delta x(\tau)$ satisfying $\delta x(\tau_1) = \delta x(\tau_2) = 0$, where τ_1 and τ_2 are the initial and final values of the parameter τ along the path, respectively.

Consider $s = \int d\tau (g_{ij}(x)\dot{x}^i\dot{x}^j)^{1/2}$. Varying with respect to $x(\tau)$,

$$0 = \delta s = \int d\tau \, 1/2 \, (g_{mn}(x)\dot{x}^m\dot{x}^n)^{-1/2} \left[\delta\dot{x}^i \, \dot{x}^j \, g_{ij} + \dot{x}^i \, \delta\dot{x}^j \, g_{ij} + \partial_k g_{ij} \, \delta x^k \, \dot{x}^i\dot{x}^j\right].$$

Notice that the first two terms in brackets are identical due to the symmetry of the metric $g_{ij} = g_{ji}$.

Now we choose an **affine parametrization** of the path, for which $g_{mn}(x)\dot{x}^m\dot{x}^n = 1$, and integrate the terms involving $\delta\dot{x}$ by parts using the fact that the metric is symmetric in exchange of its indices:

$$\begin{aligned}
\delta s &= \int d\tau \, \delta x^i \left[-\frac{d}{d\tau}(\dot{x}^j g_{ij}) + 1/2 \, \partial_i g_{kj} \, \dot{x}^k\dot{x}^j\right] \\
&= \int d\tau \, \delta x^i \left[-\ddot{x}^j \, g_{ij} - \dot{x}^j \partial_k g_{ij} \, \dot{x}^k + \frac{1}{2} \, \partial_i g_{kj} \, \dot{x}^k\dot{x}^j\right] \\
&= -\int d\tau \, \delta x^i \left[\ddot{x}^j g_{ij} + 1/2 \dot{x}^j \dot{x}^k \left(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}\right)\right]
\end{aligned}$$

Setting $\delta s = 0$, we then have,

$$\ddot{x}^j g_{ij} + \frac{1}{2}\dot{x}^j\dot{x}^k \left(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}\right) = 0$$

Multiplying by the inverse metric g^{il}, we have,

$$\begin{aligned}
0 &= \ddot{x}^l + \dot{x}^j\dot{x}^k \, \frac{1}{2} \, g^{il} \left(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}\right) \\
&= \ddot{x}^l + \dot{x}^j\dot{x}^k \, \Gamma^l_{jk},
\end{aligned}$$

where we have defined the Christoffel symbol,

$$\Gamma^l_{jk} \equiv 1/2 g^{li} \left(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}\right).$$

We have again derived the geodesic equation, this time directly in an arbitrary coordinate basis. It is important to keep in mind that the Christoffel symbol does *not* transform as a tensor, as you can easily check. Also note its symmetry property: $\Gamma^i_{jk} = \Gamma^i_{kj}$.

Covariant derivatives

The Christoffel symbol defined above plays an important role in that it allows us to define a covariant derivative. The covariant derivative acting on an arbitrary tensor is defined in an arbitrary coordinate system by,

$$\begin{aligned}\nabla_k T^{j_1\cdots j_n}_{i_1\cdots i_m} &= \partial_k T^{j_1\cdots j_n}_{i_1\cdots i_m} - \Gamma^l_{ki_1} T^{j_1\cdots j_n}_{l\cdots i_m} - \cdots - \Gamma^l_{kj_p} T^{j_1\cdots j_n}_{i_1\cdots l\cdots i_m} - \cdots \\ &\quad + \Gamma^{j_1}_{kl} T^{l\cdots j_n}_{i_1\cdots i_m} + \cdots + \Gamma^{j_p}_{kl} T^{j_1\cdots l\cdots j_n}_{i_1\cdots i_m} + \cdots\end{aligned}$$

Each covariant index gets a term in which that index appears in a Christoffel symbol with a minus sign, and each contravariant index appears similarly with a positive sign. Aside from the signs, if you think about it there is only one way to contract indices so that the equation makes sense so it should be easy to reconstruct.

The covariant derivative satisfies the following properties:

- Linearity: $\nabla_i(\alpha A + \beta B) = \alpha \nabla_i A + \beta \nabla_i B$, for any numbers $\alpha,\ \beta$ and tensors $A,\ B$
- Leibnitz rule: $\nabla_i(AB) = (\nabla_i A)B + A(\nabla_i B)$
- Commutes with contraction: $\nabla_k(A^{i_1 i_2} g_{i_1 i_2}) = \nabla_k A^i_i$
- Torsion free: $\nabla_k \nabla_j f = \nabla_j \nabla_k f$ for any scalar field f
- $\nabla_i g_{jk} = 0$

There is a simpler way to write the covariant divergence of a vector:

$$\nabla_i V^i = \partial_i V^i + \Gamma^i_{ij} V^j = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g} V^i).$$

The covariant derivative of a scalar is the ordinary derivative, $\nabla_i \phi = \partial_i \phi$. Then the Laplacian can be written simply as,

$$\nabla^2 \phi = \nabla_i (g^{ji} \nabla_j \phi) = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j \phi).$$

This allows us to easily write the Laplacian in any coordinate system, and is a useful mnemonic.

Curvature tensor

A nonvanishing curvature in a space is reflected by the fact that a vector parallelly propagated around a closed loop will not generally return to itself. As a consequence, parallel propagation of vectors successively in different directions do not commute. We don't have time to develop the theory behind curvature, but in more than two dimensions, the commutator of covariant derivatives of one-forms (which we have not discussed) can be expressed in terms of a 4-index curvature tensor, also called the **Riemann tensor**, $R_{ijk}{}^{l}$. It is often written with three lower indices and one upper index because that is how it is naturally defined, but indices can be lowered and raised with the metric and its inverse as usual.

In a coordinate system the curvature can be expressed as,

$$R_{ijk}{}^{l} = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}.$$

Properties:

- $R_{ijk}{}^{l} = -R_{jik}{}^{l}$
- $R_{[ijk]}{}^{l} = 0$
- $R_{ijkl} = -R_{ijlk}$
- $\nabla_{[i} R_{jk]l}{}^{m} = 0$

The square brackets [ ] are used to signify antisymmetrization over the indices. Similarly, parentheses ( ) will be used to signify symmetrization of indices. We will define this notation precisely (*i.e.* with coefficients) later if we use them in calculations.

Contractions of the curvature tensor play an important role in GR, so they also have names.
$R_{ik} \equiv R_{ijk}{}^{j}$ is called the Ricci tensor, and has the property $R_{ik} = R_{ki}$.
$R \equiv R^i_i = R_{ijk}{}^{j} g^{ik}$ is called the curvature scalar.

Notice that the curvature tensor is proportional to derivatives of the metric, so it vanishes in flat space: $R_{ijk}{}^{l}=0$.

Spacetime

Minkowski demonstrated in 1909 that the Lorentz transformations can be thought of as rotations in a spacetime geometry, and that Maxwell's equations of electromagnetism take a simple form in the language of spacetime. (We will review the covariant form of Maxwell's equations later in the course.) Unlike spatial manifolds we are used to, spacetime has the peculiar property that lengths as determined by the metric are not positive definite. Path lengths in spacetime are called **proper times**. Geodesics are defined as the paths which make the proper time stationary under a change of path. The metric of $(d+1)$-dimensional Minkowski spacetime (which we will also refer to as Minkowski space) can be written,

$$ds^2 = -c^2\,dt^2 + dx_d^2,$$

where $dx_d^2 \equiv \sum_{i=1}^{d}(dx^i)^2$, where c is the speed of light. For convenience we will usually define $c=1$. You will always be able to restore c into equations by dimensional analysis. Another way to say this is that c is providing the units by which we measure velocities, so a speed of .5 is to be understood as half the speed of light, and a length of .5 years is to be understood as (.5 years)$\times c$. The metric of flat spacetime diag(-1,1,1,1) is called the **Minkowski metric**.

Different authors use different choices of **signature** of the metric. The signature of a diagonal metric is simply the number of positive and negative signs in the metric. You can define the proper time as $ds^2 = -dt^2 + \mathbf{dx}^2$ or $ds^2 = dt^2 - \mathbf{dx}^2$, but it is important to be consistent. In particular, the 4-index Riemann tensor and the curvature scalar as defined above are invariant under a change of signature, but the Ricci tensor as defined above changes sign.

The expressions in previous sections for the Christoffel symbols, the curvature tensor, etc., carry through to spacetime, except that $g \equiv \det(g_{\mu\nu})$ is replaced by $|g|$. In Minkowski spacetime, all derivatives of the metric vanish in Minkowski coordinates, so the Christoffel symbols and the curvature all vanish.

The Laplacian in spacetime is called the d'Alembertian. In Minkowski spacetime,

$$\Box\phi = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j\phi) = (-\partial_t^2 + \nabla_d^2)\phi.$$

Einstein's equations

General relativity describes the dynamics of the geometry of spacetime. This dynamics is described by Einstein's equations,

$$R_{\mu\nu} - 1/2\, g_{\mu\nu}R = 8\pi G_N\, T_{\mu\nu},$$

where $T_{\mu\nu}$ is the **stress tensor**, and the combination $R_{\mu\nu} - 1/2\, g_{\mu\nu}R$ is called the **Einstein tensor**. G_N is called Newton's constant. The stress tensor is covariantly conserved:

$$\nabla_\mu T^{\mu\nu} = 0.$$

For consistency of Einstein's equations this requires that,

$$\nabla_\mu(R^{\mu\nu} - 1/2 g^{\mu\nu}R) = 0.$$

Writing out the Einstein tensor in terms of the metric, one can show that it is in fact an identity that its covariant divergence vanishes. In fact, one line of arguments leading to Einstein's equations is that the stress tensor should appear as the source of gravity, and the simplest generally covariant, conserved rank-2 tensor made from derivatives of the metric is the Einstein tensor, so it should appear in Einstein's equations. Alternatively, one can start with the action,

$$S = \int d^{d+1}x\sqrt{|g|}(R/G + \mathcal{L}_{\text{matter}})$$

Varying S with respect to the metric gives Einstein's equations, where the stress tensor is,

$$T^{\mu\nu} = -\frac{1}{8\pi\sqrt{|g|}}\frac{\delta(\sqrt{|g|}\mathcal{L}_{\text{matter}})}{\delta g_{\mu\nu}}.$$

Notice that in general relativity the geometry of spacetime appears through the Ricci tensor and the curvature scalar, but does not involve the curvature tensor in its full glory. Hence, solutions to the equations of general relativity are not unique and require certain boundary conditions to specify the solutions completely. For example, in vacuum ($T_{ij} = 0$) three different solutions to Einstein's equations in 3+1 dimensions are:

- flat space: $ds^2 = -dt^2 + \mathbf{dx}^2$
- Black hole: $ds^2 = -\left(1 - \frac{2G_N M}{r}\right)^{-1} dt^2 + \left(1 - \frac{2G_N M}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$
- Gravity waves: we will discuss these later

Also notice that the Einstein tensor as defined above changes sign with a change in signature of the metric, as does the stress tensor. In reading the literature you should also note that sometimes people define the stress tensor with the opposite sign to our definition. You are thereby cautioned to be careful when comparing equations coming from GR in different references.

All of the equations we have written so far are valid in any number of dimensions > 2.

References

[1] Robert M. Wald, General Relativity, ch. 3&4.