PHYS 630 S'24 Problem Set 4 Solutions

Kardar, Ch. 2

11. a) $M(x,y) \equiv \sum_{x,y} p(x,y) \ln\left(\frac{p(x,y)}{p_x(x)\, p_y(y)}\right)$

$$= \sum_{x,y} p(x,y) \ln p(x,y) - \sum_x \left(\sum_y p(x,y)\right) \ln p_x(x)$$

$$- \sum_y \left(\sum_x p(x,y)\right) \ln p_y(y)$$

$$= \underbrace{\sum_{x,y} p(x,y) \ln p(x,y)}_{} \underbrace{- \sum_x p_x(x) \ln p_x(x)}_{} \underbrace{- \sum_y p_y(y) \ln p_y(y)}_{}$$

$$= \quad -S(x,y) \quad + \quad S(x) \quad + \quad S(y)$$

Hence, $\boxed{M(x,y) = -S(x,y) + S(x) + S(y)}$

b) $p(x,y) = \mathcal{N} \exp\left(-\frac{a x^2}{2} - \frac{b y^2}{2} - cxy\right)$

Normalization

Fix $\mathcal{N}$ with $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; p(x,y) = 1$

$$1 = \mathcal{N} \iint_{-\infty}^{\infty} dx\, dy \; \exp\left(-\frac{a}{2}\left(x + \frac{c y}{a}\right)^2 - \frac{1}{2}\left(b - \frac{c^2}{a}\right) y^2\right)$$

Change variables: $x' = \left(x + \frac{cy}{a}\right)$, $y' = y$

Jacobian $(x \to x')$ $J^{-1} = \det\begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \det\begin{pmatrix} 1 & \frac{c}{a} \\ 0 & 1 \end{pmatrix} = 1$

$$1 = \mathcal{N} \iint_{-\infty}^{\infty} dx'\,dy' \exp\left(-\frac{a}{2}x'^2 - \frac{1}{2}\left(b - \frac{c^2}{a}\right)y^2\right)$$

$$= \mathcal{N} \left(\int_{-\infty}^{\infty} dx' \exp\left(-\frac{a}{2}x'^2\right)\right)$$

$$\times \left(\int_{-\infty}^{\infty} dy' \exp\left(-\frac{1}{2}\left(b - \frac{c^2}{a}\right)y^2\right)\right)$$

$$= \mathcal{N} \sqrt{\frac{2\pi}{a}} \cdot \sqrt{\frac{2\pi}{(b - c^2/a)}}$$

$$\Rightarrow \mathcal{N} = \frac{1}{2\pi}\left(ab - c^2\right)^{1/2}$$

Then $P_y(y) = \int_{-\infty}^{\infty} dx\, p(x,y)$

$$= \frac{1}{2\pi}(ab - c^2)^{1/2}\left(\frac{2\pi}{a}\right)^{1/2} \exp\left(-\frac{1}{2}\left(b - \frac{c^2}{a}\right)y^2\right)$$

Completing the square as above

$$P_y(y) = \left[\frac{1}{2\pi}\left(b - \frac{c^2}{a}\right)\right]^{1/2} \exp\left(-\left(\frac{b}{2} - \frac{c^2}{2a}\right)y^2\right)$$

By exchanging $(a \leftrightarrow b)$, $(x \leftrightarrow y)$:

$$P_x(x) = \left[\frac{1}{2\pi}\left(a - \frac{c^2}{b}\right)\right]^{1/2} \exp\left(\left(-\frac{a}{2} - \frac{c^2}{2b}\right)x^2\right)$$

Now,

$$M(x,y) = \iint_{-\infty}^{\infty} dx\, dy\; p(x,y) \left[\ln \frac{\frac{1}{2\pi}(ab-c^2)^{1/2}}{\frac{1}{2\pi}\left(b-\frac{c^2}{a}\right)^{1/2}\left(a-\frac{c^2}{b}\right)^{1/2}} \right.$$

$$+ \left(-\frac{ax^2}{2} - \frac{by^2}{2} - cxy\right)$$

$$\left. + \left(\frac{b}{2} - \frac{c^2}{2a}\right)y^2 + \left(\frac{a}{2} - \frac{c^2}{2b}\right)x^2 \right]$$

$$= \iint_{-\infty}^{\infty} dx\, dy\; p(x,y) \left[\ln\left(\frac{\cancel{(ab-c^2)}}{\cancel{(ab-c^2)}\left(1-\frac{c^2}{ab}\right)}\right)^{1/2} \right.$$

$$\left. - cxy - \frac{c^2}{2a}y^2 - \frac{c^2}{2b}x^2 \right]$$

Note! $-\frac{c}{2}xy = -\frac{c}{2}\left(x+\frac{cy}{a}\right)y + \frac{c^2y^2}{2a}$

$$= -\frac{c}{2}x'y' + \frac{c^2y'^2}{2a},$$

with $x' = x + \frac{cy}{a}$ as earlier.

Hence, $\iint_{-\infty}^{\infty} dx\,dy \left(-\frac{c}{2} xy + \frac{c^2}{2a} y^2\right) p(x,y)$

$$= \iint_{-\infty}^{\infty} dx'\,dy' \left(-\frac{c}{2} x'y'\right) p(x(x'), y(y'))$$

$= 0$ because $x'y'$ is odd under $x' \to -x'$, while $p(x(x'), y(y'))$ is even.

Similarly,

$$\iint_{-\infty}^{\infty} dx\,dy \left(-\frac{c}{2} xy + \frac{c^2}{2b} x^2\right) p(x,y) = 0.$$

Finally, using $\iint dx\,dy\, p(x,y) = 1$,

$$\boxed{M(x,y) = -\frac{1}{2} \ln\left(1 - \frac{c^2}{ab}\right)}$$

Kardar, Ch. 3

1. a) 1-D trap: $y = z = 0$, $x \equiv q$

$$\rho(q, p, t=0) = \delta(q) f(p),$$

$$f(p) = \frac{\exp(-p^2/2mk_BT)}{\sqrt{2\pi m k_B T}}$$

Liouville's Eq.:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial H}{\partial p}\frac{\partial \rho}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial \rho}{\partial p}$$

Here, the particles move freely in 1D: $H = \frac{p^2}{2m}$

$$\rightarrow \left.\frac{\partial \rho}{\partial t}\right|_{q,p} = -\frac{p}{m}\left.\frac{\partial \rho}{\partial q}\right|_{p,t}$$

Since p is held fixed in this eq., solutions are of the form

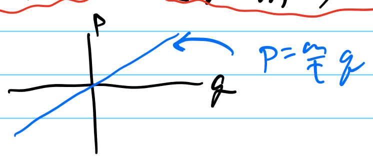
$$\rho = \rho\left(q - \frac{p}{m}t, p\right).$$

(Check using the chain rule in Liouville's Eq.)

Intuitively, all of the particles with momentum p move with velocity p/m.

Given $\rho(q, p, t=0) = \delta(q) f(p)$, the solution to Liouville's eq. is

$$\boxed{\rho(q, p, t) = \delta\left(q - \frac{p}{m}t\right) f(p)}$$



b) $\langle p^2 \rangle = \iint_{-\infty}^{\infty} dq\, dp\; p^2 f(p)\, \delta\!\left(q - \frac{p}{m} t\right)$

$$= \int_{-\infty}^{\infty} dp\; p^2 f(p)$$

$$= \int_{-\infty}^{\infty} dp\; p^2 \frac{\exp\left(-\frac{p^2}{2mk_B T}\right)}{\sqrt{2\pi m k_B T}} = \boxed{m k_B T}$$

$$\langle q^2 \rangle = \iint_{-\infty}^{\infty} dq\, dp\; q^2 f(p)\, \delta\!\left(q - \frac{p}{m} t\right)$$

$$= \int_{-\infty}^{\infty} dp \left(\frac{p}{m} t\right)^2 \frac{\exp\left(-\frac{p^2}{2mk_B T}\right)}{\sqrt{2\pi m k_B T}} = m k_B T \left(\frac{t^2}{m^2}\right)$$

$$= \boxed{\frac{k_B T}{m} t^2}$$