PHYS 630 S'24 Problem Set 10 Solutions

Kardar 6.6

a) $\rho = \dfrac{\exp(-\beta \mathcal{H})}{\mathrm{Tr}\, e^{-\beta\mathcal{H}}}$, $\mathcal{H} = -\mu_B \vec{\sigma}\cdot\vec{B}$, $\vec{B} = B\hat{z}$

Note: $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$.

$$\exp(\beta\mu_B B\sigma_z) = \left(1 + \tfrac{1}{2}(\beta\mu_B B)^2 + \tfrac{1}{4!}(\beta\mu_B B)^4 + \dots\right)\mathbb{1}$$
$$+ \left((\beta\mu_B B) + \tfrac{1}{3!}(\beta\mu_B B)^3 + \dots\right)\sigma_z$$
$$= \cosh(\beta\mu_B B)\,\mathbb{1} + \sinh(\beta\mu_B B)\,\sigma_z$$

$$\mathrm{Tr}\, e^{-\beta\mathcal{H}} = {}_z\langle\uparrow| e^{-\beta\mathcal{H}} |\uparrow\rangle_z + {}_z\langle\downarrow| e^{-\beta\mathcal{H}} |\downarrow\rangle_z$$
$$= \cosh(\beta\mu_B B)\,\mathrm{Tr}\,\mathbb{1}_{2\times 2} + \sinh(\beta\mu_B B)\,\cancelto{0}{\mathrm{Tr}\,\sigma_z}$$
$$= 2\cosh(\beta\mu_B B)$$

Hence, $\boxed{\rho} = \dfrac{\exp(\beta\mu_B B\sigma_z)}{\mathrm{Tr}\, e^{-\beta\mathcal{H}}}$ $\boxed{= \dfrac{1}{2\cosh(\beta\mu_B B)}\begin{pmatrix} e^{\beta\mu_B B} & 0 \\ 0 & e^{-\beta\mu_B B}\end{pmatrix}}$

Average energy: $\boxed{\langle\mathcal{H}\rangle} = \mathrm{Tr}\,\mathcal{H}\rho = \dfrac{\mathrm{Tr}\left[-\mu_B B\sigma_z \exp(\beta\mu_B B\sigma_z)\right]}{2\cosh(\beta\mu_B B)}$

$$= -\mu_B B\,\frac{\sinh(\beta\mu_B B)}{\cosh(\beta\mu_B B)} \boxed{= -\mu_B B\tanh(\beta\mu_B B)}$$

7.1 Bosons : $|k_1, k_2\rangle_+ = \begin{cases} \dfrac{|k_1\rangle|k_2\rangle + |k_2\rangle|k_1\rangle}{\sqrt{2}} & k_1 \neq k_2 \\ |k_1 k_2\rangle & k_1 = k_2 \end{cases}$

$$\boxed{Z_2^+} = \mathrm{Tr}\, e^{-\beta\mathcal{H}} = \sum_{k_1 > k_2} {}_+\langle k_1, k_2| e^{-\beta\mathcal{H}} |k_1, k_2\rangle_+ + \sum_{k} \langle k, k| e^{-\beta\mathcal{H}} |k k\rangle$$

$$= \sum_{k_1 > k_2} \exp\left[-\frac{\beta\hbar^2}{2m}(k_1^2 + k_2^2)\right] + \sum_k \exp\left(-\frac{2\beta\hbar^2}{2m}k^2\right)$$

$$= \frac{1}{2}\sum_{k_1, k_2} \exp\left[-\frac{\beta\hbar^2}{2m}(k_1^2 + k_2^2)\right] + \frac{1}{2}\sum_k \exp\left(-\frac{2\beta\hbar^2 k^2}{2m}\right)$$

includes $k_1 = k_2$, $k_1 < k_2$

other $\frac{1}{2}$ the sum is included in the first term.

$$\boxed{= \frac{1}{2}\left\{Z_1(m)^2 + Z_1(m/2)\right\}}$$

Fermions: $|k_1, k_2\rangle_- = \dfrac{|k_1\rangle|k_2\rangle - |k_2\rangle|k_1\rangle}{\sqrt{2}}$

$$Z_2^- = \mathrm{Tr}\, e^{-\beta\mathcal{H}} = \sum_{k_1,k_2} {}_-\langle k_1, k_2| e^{-\beta\mathcal{H}} |k_1, k_2\rangle_-$$

$$= \sum_{k_1>k_2} \exp\left[-\frac{\beta\hbar^2}{2m}(k_1^2 + k_2^2)\right]$$

$$= \frac{1}{2}\sum_{k_1,k_2} \exp\left[-\frac{\beta\hbar^2}{2m}(k_1^2+k_2^2)\right] - \frac{1}{2}\sum_{k} \exp\left(-\frac{\beta\hbar^2 k^2}{2m}\right)$$

includes $k_1 = k_2$, $k_1 < k_2$

$$= \frac{1}{2}\left[Z_1(m)^2 - Z_1(m/2)\right]$$

b) $Z_1(m) = V/\lambda^3$, $\lambda = h/\sqrt{2\pi m k_B T}$

$$\ln Z_2^{\pm} = \ln\left\{\left[Z_1(m)^2 \pm Z_1(m/2)\right]/2\right\}$$

$$= 2\ln Z_1(m) + \ln\left[1 \pm \frac{Z_1(m/2)}{Z_1(m)^2}\right] - \ln 2$$

$\ll 1$ if $V/\lambda^3 \gg 1$

Use $\ln(1+x) \approx x$ if $x \ll 1$

$$\approx \underbrace{2\ln Z_1(m) - \ln 2}_{\text{classical result}} \pm \frac{Z_1(m/2)}{Z_1(m)^2}$$

$$\ln Z_2^{\pm} \approx \ln Z_2^{classical} \pm \left(V/\lambda(m/2)^3\right)\left(V/\lambda(m)^3\right)^{-2}$$

$$= \ln Z_2^{classical} \pm 2^{-3/2} \frac{\lambda(m)^3}{V}$$

$$= \ln Z_2^{classical} \underbrace{\pm 2^{-3/2} \frac{h^3}{V(2\pi m k_B T)^{3/2}}}_{\Delta \ln Z_2^{\pm}}$$

Correction to classical energy:

$\lambda^3 \propto T^{-3/2}$

$$\boxed{\Delta E^{\pm}} = -\frac{\partial}{\partial \beta}\Delta \ln Z_2^{\pm} \boxed{= \mp 2^{-3/2} \cdot \frac{3}{2} \frac{\lambda^3}{V}(k_B T)}$$

Correction to classical heat capacity:

$$\boxed{\Delta C_V^{\pm}} = \frac{d\Delta E}{dT}\Big|_V = \mp 2^{-3/2} \cdot 3/2 \cdot \left(-\frac{1}{2}\right)\frac{\lambda^3}{V} k_B$$

$$\boxed{= \pm 3 \cdot 2^{-7/2} \frac{\lambda^3}{V} k_B}$$

c) Approximation breaks down if $\lambda^3 \gtrsim V$

$$\rightarrow \frac{h^3}{(2\pi m k_B T)^{3/2}} \gtrsim V$$

$$\rightarrow \boxed{T \lesssim \frac{h^2}{2\pi m k_B V^{2/3}}}$$