Degenerate Bose Gas

Avg. bose occupation number:

$$\langle n_{\vec{k}} \rangle_+ = \frac{1}{\exp\{\beta(\mathcal{E}(\vec{k}) - \mu)\} - 1}$$

$$\langle n_{\vec{k}} \rangle_+ \geq 0 \rightarrow \mu < \min(\mathcal{E}(\vec{k})) = 0$$

High-T, low density: $\mu \to -\infty$, $z \to 0$

Low-T, high density: $\mu \to 0$, $z \to 1$

From the general discussion of the nonrelativistic quantum gas:

$$\begin{cases} \beta P_+ = \frac{g}{\lambda^3} f^+_{5/2}(z) \\ n_+ = \frac{g}{\lambda^3} f^+_{3/2}(z) \\ \mathcal{E}_+ = \frac{3}{2} P_+ \end{cases}$$

where $f^+_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - 1}$

As $z \to 0$,

$$\frac{n_+ \lambda^3}{g} = f^+_{3/2}(z) \rightarrow z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \cdots$$

$$\rightarrow z = \frac{n_+ \lambda^3}{g} - \frac{1}{2^{3/2}}\left(\frac{n_+ \lambda^3}{g}\right)^2 - \cdots$$

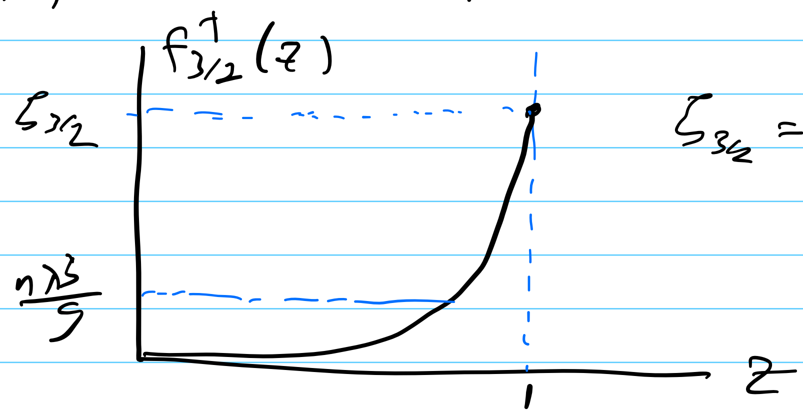
High-T, $z \ll 1$:

$$\frac{\beta P_+ \lambda^3}{g} = f^+_{5/2}(z) = z + \frac{z^2}{2^{5/2}} + \cdots$$

$$= \frac{n_+ \lambda^3}{g} + \left(\frac{1}{2^{5/2}} - \frac{1}{2^{3/2}}\right)\left(\frac{n_+ \lambda^3}{g}\right)^2 + \cdots$$

Low-T, $z \to 1$ Degenerate Bose gas.

$f^+_m(z)$ increases w/ z in range $z \in [0, 1]$



$\zeta_{3/2} = f^+_{3/2}(1) = 2.612\ldots$

Define $\zeta_m = f^+_m(1) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}}{e^x - 1}\, dx$

Integrand $\sim x^{m-2}$ as $x \to 0$

$\rightarrow \zeta_m$ finite for $m > 1$
infinite for $m \leq 1$.

Can show $\frac{d}{dz} f^+_m(z) = \frac{1}{z} f^+_{m-1}(z)$

$\rightarrow$ For any m, a sufficiently high derivative of $f^+_m(z)$ will diverge at $z = 1$.

Density of excited states

$$n_x = \frac{g}{\lambda^3} f^+_{3/2}(z)$$

[Ground state is populated in $\langle n_{\vec{k}} \rangle = \frac{1}{e^{\beta(\varepsilon(\vec{k}) - \mu)} - 1}$, but $\vec{k} = \vec{0}$ mode must be included separately in the thermodynamic approximation $N = \sum_{\vec{k}} \langle n_{\vec{k}} \rangle \approx V \int \frac{dk}{(2\pi)^3} \frac{4\pi k^2}{e^{\beta(\varepsilon - \mu)} - 1}$]

$$n_x \leq n^* = \frac{g}{\lambda^3} \zeta_{3/2} \quad \propto T^{3/2}$$

At high T, $\frac{n\lambda^3}{g} = \frac{n}{g}\left(\frac{h}{\sqrt{2\pi m k_B T}}\right)^3 \ll \zeta_{3/2} = 2.612\ldots$

$$n_x \approx n$$

Lowering T, the limiting density of excited state is reached when

$$\frac{n \lambda(T_c)^3}{g} = \zeta_{3/2}$$

$$\rightarrow T = T_c(n) = \frac{h^2}{2\pi m k_B}\left(\frac{n}{g\zeta_{3/2}}\right)^{2/3}$$

Bose-Einstein Condensation: $T < T_c$

For all $T < T_c$, $z \approx 1$, i.e. $\mu \approx 0$.

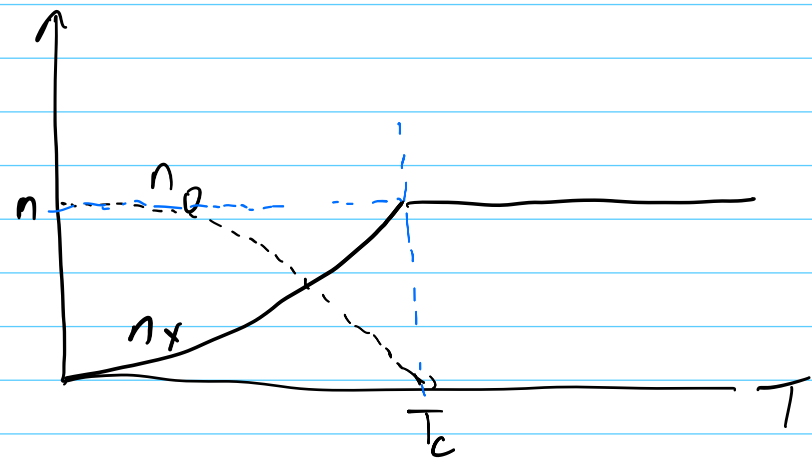
Density of excited states $n_x = n^* = \frac{g\zeta_{3/2}}{\lambda^3} \propto T^{3/2}$

$$< n$$

$\rightarrow$ $n_0 = n - n_x$ = density of gas particles in $\vec{k} = \vec{0}$ ground state.

Macroscopic occupation of lowest-energy state

= Bose-Einstein Condensation.



$T=0$: $n_x = n^* = 0 \quad \rightarrow \quad n_0 = n$

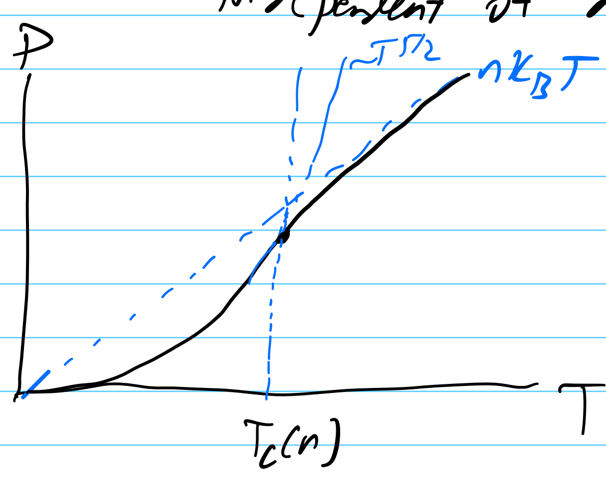
All gas particles in ground state.

<u>Gas pressure</u> for $T < T_c$:

$$\beta P_+ = \frac{g}{\lambda^3} f^+_{5/2}(1) = \frac{g}{\lambda^3} \zeta_{5/2} \approx 1.341 \frac{g}{\lambda^3}$$

$$P_+ \propto T^{5/2} \quad \text{as } T \to 0.$$

– independent of density n.



Compressibility:

$$K_T = \frac{1}{n} \frac{\partial n}{\partial P}\bigg|_T$$

Use $\frac{dP}{dz} = \frac{g k_B}{\lambda^3} \frac{1}{z} f^+_{3/2}(z)$

$$\frac{dn}{dz} = \frac{g}{\lambda^3} \frac{1}{z} f^+_{1/2}(z)$$

$$\rightarrow K_T = \frac{1}{n} \frac{dn}{dz} \Big/ \frac{dP}{dz}$$

$$= \frac{f^+_{1/2}(z)}{n k_B T f^+_{3/2}(z)}$$

$f^+_{1/2}(z) \rightarrow \infty$ as $z \rightarrow 1$, so

$K_T \rightarrow \infty$ as $T \rightarrow T_C$ from above.

Heat capacity

$$E = \frac{3}{2} PV = \frac{3}{2} V \frac{g}{\lambda^3} k_B T f^+_{5/2}(z)$$

$$\propto T^{5/2} f^+_{5/2}(z)$$

$$C_V = \frac{dE}{dT}\bigg|_{V,N} = \frac{3}{2} V \frac{g}{\lambda^3} k_B T \left[\frac{5}{2T} f^+_{5/2}(z) + \frac{1}{z} f^+_{3/2}(z) \frac{dz}{dT}\bigg|_{V,N} \right]$$

To find $\left.\frac{dz}{dT}\right|_{V,N}$, use

$$\left.\frac{dN}{dT}\right|_V = 0 = \frac{g}{\lambda^3} V \left[\frac{3}{2T} f^+_{3/2}(z) + \frac{1}{z} f^+_{1/2}(z) \left.\frac{dz}{dT}\right|_{V,N} \right]$$

$$\rightarrow \left.\frac{T}{z}\frac{dz}{dT}\right|_{V,N} = -\frac{3}{2}\frac{f^+_{3/2}(z)}{f^+_{1/2}(z)}$$

Then, $C_V = \frac{3}{2} V \frac{g}{\lambda^3} k_B \left[\frac{5}{2} f^+_{5/2}(z) - \frac{3}{2} \frac{f^+_{3/2}(z)^2}{f^+_{1/2}(z)} \right]$

High-T, low n: $z \ll 1$

$$\frac{C_V}{Nk_B} = \frac{3}{2}\left[1 + n\frac{\lambda^3}{2^{7/2}} + \cdots\right] > \frac{3}{2}$$

classical value. $T \to \infty$

Low-T : $z = 1$

$$\frac{C_V}{Nk_B} = \frac{3}{2}\frac{g}{n\lambda^3} k_B \left[\frac{5}{2}\zeta_{5/2}(1) - \frac{3}{2}\frac{\zeta_{3/2}(1)}{\zeta_{1/2}(1)} \right]$$

$= 0$ because $\zeta_{1/2}(1) = \infty$.

$$= \frac{15}{4}\frac{g}{n\lambda^3}\zeta_{5/2}$$

$$= \frac{15}{4}\zeta_{5/2} \cdot \frac{1}{\zeta_{3/2}}\left(\frac{T}{T_c}\right)^{3/2}$$

$$\leftarrow \frac{n\lambda(T_c)^3}{g} = \zeta_{3/2}$$

Hence, $C_V \propto T^{3/2}$ for $T < T_c$