Kardar 7.4

Non-relativistic Quantum Gases

Including a spin degeneracy factor $g = 2s+1$, the grand partition function for the quantum gas takes the form

$$\mathcal{Q}_\eta = \prod_{\vec{k}} \left[1 - \eta \exp(\beta\mu - \beta\mathcal{E}(\vec{k}))\right]^{-\eta g}$$

$$\text{where } \mathcal{E}(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m}, \quad \eta = \begin{cases} +1 & \text{bosons} \\ -1 & \text{fermions} \end{cases}$$

$$\ln \mathcal{Q}_\eta = -\eta g \sum_{\vec{k}} \ln\left[1 - \eta \exp(\beta\mu - \beta\mathcal{E}(\vec{k}))\right]$$

$$= -\eta g V \int \frac{d^3k}{(2\pi)^3} \ln\left[1 - \eta z \exp\left(-\frac{\beta\hbar^2 k^2}{2m}\right)\right]$$

$$\text{where } z = e^{\beta\mu}.$$

Hence,

$$\boxed{\beta P_\eta} = \frac{\ln \mathcal{Q}_\eta}{V} \boxed{= -\eta g \int \frac{d^3k}{(2\pi)^3} \ln\left[1 - \eta z \exp\left(-\frac{\beta\hbar^2 k^2}{2m}\right)\right]}$$

$$\boxed{n_\eta} \equiv \frac{N_\eta}{V} = \frac{g}{V}\sum_{\vec{k}} \langle n_{\vec{k}} \rangle$$

$$\boxed{= g \int \frac{d^3k}{(2\pi)^3} \frac{1}{z^{-1}\exp\left(\frac{\beta\hbar^2 k^2}{2m}\right) - \eta}}$$

$$\boxed{\mathcal{E}_\eta} \equiv \frac{E_\eta}{V} = \boxed{g\int \frac{d^3k}{(2\pi)^3}\,\frac{\hbar^2 k^2}{2m}\,\frac{1}{z^{-1}\exp\left(\frac{\beta\hbar^2 k^2}{2m}\right)-\eta}}$$

Change variables $\quad x \equiv \dfrac{\beta\hbar^2 k^2}{2m}$

$$\rightarrow k = \sqrt{\frac{2mk_BT\,x}{\hbar^2}} = \frac{2\sqrt{\pi}}{\lambda}\,x^{1/2}$$

$$dk = \frac{\sqrt{\pi}}{\lambda}\,x^{-1/2}\,dx$$

$$\rightarrow \boxed{\beta P_\eta} = -\eta\,\frac{g}{2\pi^2}\,\frac{4\pi^{3/2}}{\lambda^3}\int_0^\infty dx\,x^{1/2}\ln\left(1-\eta z e^{-x}\right)$$

by parts

$$= +\eta\,\frac{g}{2\pi^2}\,\frac{4\pi^{3/2}}{\lambda^3}\int_0^\infty dx\left(\frac{2}{3}x^{3/2}\right)\cdot\frac{\eta z e^{-x}}{1-\eta z e^{-x}}$$

$$\boxed{= \frac{g}{\lambda^3}\,\frac{4}{3\sqrt{\pi}}\int_0^\infty dx\,\frac{x^{3/2}}{z^{-1}e^{x}-\eta}}$$

$$\boxed{n_\eta = \frac{g}{\lambda^3}\,\frac{2}{\sqrt{\pi}}\int_0^\infty dx\,\frac{x^{1/2}}{z^{-1}e^{x}-\eta}}$$

$$\boxed{\mathcal{E}_\eta = \frac{g}{\lambda^3\beta}\,\frac{2}{\sqrt{\pi}}\int_0^\infty dx\,\frac{x^{3/2}}{z^{-1}e^{x}-\eta}}$$

Functions of the form

$$f_m^\eta(z) \equiv \frac{1}{\Gamma(m)} \int_0^\infty \frac{dx\ x^{m-1}}{z^{-1} e^x - \eta}$$

show up often. In terms of these,

$$\boxed{\beta P_\eta = \frac{g}{\lambda^3} f^\eta_{5/2}(z)}$$

$$\boxed{n_\eta = \frac{g}{\lambda^3} f^\eta_{3/2}(z)}$$

$$\boxed{\mathcal{E}_\eta} = \frac{3g}{2\lambda^3\beta} f^\eta_{5/2}(z)$$

$$\boxed{= \frac{3}{2} P_\eta}$$

Recall,

$$\Gamma(m) = \int_0^\infty dx\, x^{m-1} e^{-x}$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(5/2) = \frac{3\sqrt{\pi}}{4}$$

High-Temperature, Low density expansions:

$z = e^{\beta\mu} \ll 1$ so that $n_\eta \frac{\lambda^3}{g} \ll 1$

Then $\boxed{f_m^\eta(z) = z + \eta \frac{z^2}{2^m} + \frac{z^3}{3^m} + \eta \frac{z^4}{4^m} + \cdots}$

$$\frac{n_\eta \lambda^3}{g} = f^\eta_{3/2}(z) \simeq z + \eta \frac{z^2}{2^{3/2}} + \cdots$$

$$\rightarrow \boxed{z \simeq \frac{n_\eta \lambda^3}{g} + \cdots}$$

Solve for z recursively.

$$\boxed{\frac{\beta P_\eta \lambda^3}{g}} = f^{\eta}_{5/2}(z) = z + \eta \frac{z^2}{2^{5/2}} + \cdots$$

$$\boxed{\approx \frac{n_\eta \lambda^3}{g} + \cdots}$$

Including the leading corrections,

$$P_\eta \approx n_\eta k_B T\left[1 - \frac{\eta}{2^{5/2}}\left(\frac{n_\eta \lambda^3}{g}\right) + \cdots\right]$$

$$\boxed{PV = N k_B T\left[1 - \frac{\eta}{2^{5/2}}\left(\frac{N \lambda^3}{V g}\right) + \cdots\right]}$$

7.5 In the high-density, low-T regime, $\frac{n\lambda^3}{g} > 1$

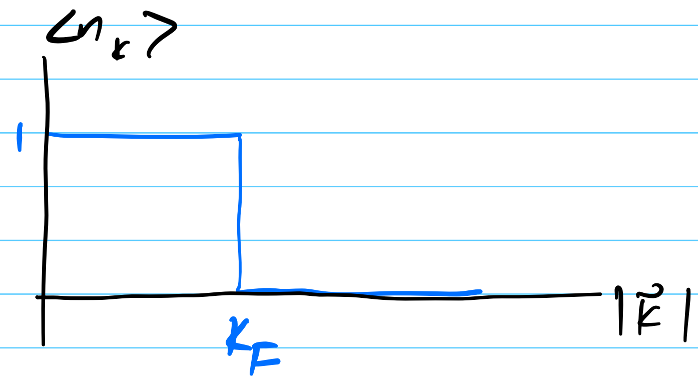
— Quantum degenerate limit

Degenerate fermi gas

Fermi occupation number:

$$\langle n_{\vec{k}} \rangle_- = \frac{1}{e^{\beta(\varepsilon(\vec{k}) - \mu)} + 1}$$

As $\beta \to \infty$, $\langle n_{\vec{k}} \rangle_- \to \begin{cases} 1 & \text{if } \varepsilon(\vec{k}) < \mu \\ 0 & \text{if } \varepsilon(\vec{k}) > \mu \end{cases}$



Fermi wavenumber k_F determined by

$$N = \sum_{|\vec{k}| \le k_F} (2s+1) = gV \int^{k<k_F} \frac{d^3k}{(2\pi)^3}$$

$$= \frac{gV}{8\pi^3} \cdot 4\pi \int_0^{k_F} k^2 \, dk$$

$$= \frac{gV}{6\pi^2} k_F^3$$

$$\rightarrow \boxed{k_F = \left(\frac{6\pi^2 n}{g} \right)^{1/3}} \quad \text{where } n = \frac{N}{V}.$$

Fermi energy: $\boxed{\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 n}{g} \right)^{2/3}}$

At $T=0$ there is a unique ground state with all single-particle states with $k < k_F$ filled.

At low but nonvanishing T, we can expand

$$\lim_{z\to\infty} f_m^-(z) = \frac{(\ln z)^m}{\Gamma(m+1)}\left[1 + \frac{\pi^2}{6}\frac{m(m-1)}{(\ln z)^2} + \cdots\right]$$

$$\frac{n\lambda^3}{g} = f_{3/2}^-(z) = \frac{(\ln z)^{3/2}}{\Gamma(5/2)}\left[1 + \frac{\pi^2}{6}\frac{3}{4(\ln z)^2} + \cdots\right]$$

$\gg 1$

$T \to 0$: $\quad \ln z = \left[\frac{3}{4\sqrt{\pi}}\frac{n\lambda^3}{g}\right]^{2/3} + \cdots$

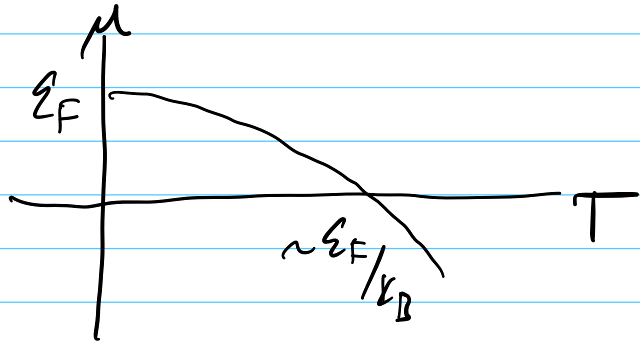
$$= \frac{\beta\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3} + \cdots$$

$$= \beta\varepsilon_F + \cdots$$

Leading correction:

$$\ln z = \beta\varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \cdots\right]$$

Chemical potential $\mu = k_B T \ln z$



$$\beta P = \frac{g}{\lambda^3} f^-_{5/2}(z) = \beta P_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{\varepsilon_F} \right)^2 + \cdots \right]$$

$P_F = \frac{2}{5} n \varepsilon_F$ Fermi Pressure.

Note: As $T \to 0$, $P \to P_F > 0$

$$\frac{E}{V} = \frac{3}{2} P = \frac{3}{5} n \varepsilon_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{\varepsilon_F} \right)^2 + \cdots \right]$$

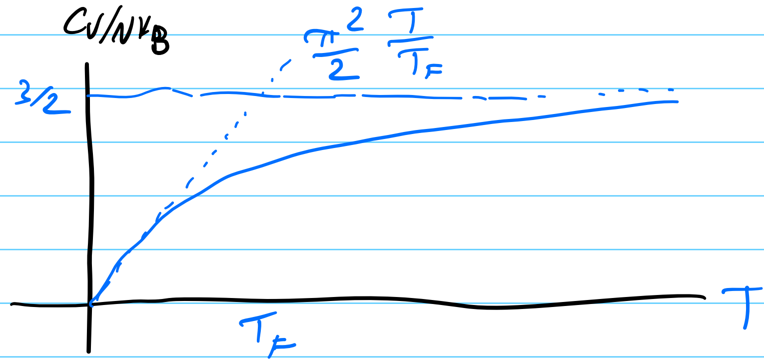
$$= \frac{3}{5} n k_B T_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{T}{T_F} \right)^2 + \cdots \right]$$

where $T_F \equiv \varepsilon_F / k_B$ Fermi Temperature.

Heat Capacity at low-T:

$$C_V = \frac{dE}{dT} = \frac{\pi^2}{2} N k_B \frac{T}{T_F} + \cdots$$

$\propto T$



C_V/Nk_B
$3/2$
$\frac{\pi^2}{2}\frac{T}{T_F}$
T_F
T