PHYS 621 Lecture Notes 9

N5.5

The Free Nonrelativistic Particle

$$\hat{H} = \frac{\hat{P}^2}{2m}$$

Eigenvalue problem: $\hat{H}|\psi_E\rangle = E|\psi_E\rangle$

$$\frac{\hat{P}^2}{2m}|\psi_E\rangle = E|\psi_E\rangle$$

$\rightsquigarrow |\psi_E\rangle = |p\rangle$ Momentum eigenstate

$$\frac{\hat{P}^2}{2m}|p\rangle = \frac{p^2}{2m}|p\rangle \rightarrow E = \frac{p^2}{2m}$$

Degeneracy: $|p\rangle$ and $|-p\rangle$ both have $E = \frac{p^2}{2m}$.

Time-dependence: $i\hbar \frac{d}{dt}|\psi_p(t)\rangle = \hat{H}|\psi_p(t)\rangle$

$$= \frac{p^2}{2m}|\psi_p(t)\rangle$$

$$|\psi_p(t)\rangle = e^{-i\frac{p^2}{2m}t/\hbar}|p\rangle$$

Stationary state: $\forall$ observables $\hat{A}$ (not explicitly depending on t)

$$\langle\psi_p(t)|\hat{A}|\psi_p(t)\rangle$$

$$= \langle p|e^{+ip^2/2m\,t/\hbar}\,\hat{A}\,e^{-ip^2/2m\,t/\hbar}|p\rangle$$

$$= \langle p|\hat{A}|p\rangle$$ — time-independent

Naively:

wave function:

$$\langle x | \psi_p(t) \rangle = e^{-i\frac{p^2}{2m\hbar}t} \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$$

$$|\psi_p(t)\rangle = \int dx\, |x\rangle \langle x | \psi_p(t) \rangle$$

If $|\psi\rangle = \int dx'\, |x'\rangle\, f(x')$

then $\text{prob}(a < x < b) = \int_a^b dx\, |\langle x | \psi \rangle|^2$

$$= \int_a^b dx \left| \int_{-\infty}^{\infty} dx' \langle x | x' \rangle f(x') \right|^2$$

$$= \int_a^b dx \left| \int_{-\infty}^{\infty} dx'\, \delta(x - x')\, f(x') \right|^2$$

$$= \int_a^b dx\, |f(x)|^2$$

Normalization: $\langle \psi | \psi \rangle = 1 \rightarrow \int dx\, |f(x)|^2 = 1.$

For the state $|\psi_p(t)\rangle$ we would have

$$\text{prob}(a < x < b) \stackrel{?}{=} \int_a^b |\langle x | \psi_p(t) \rangle|^2\, dx$$

$$\stackrel{?}{=} \int_a^b \frac{1}{2\pi\hbar}\, dx$$

The units don't even match. What happened?

The problem is that the state $|\psi_p(t)\rangle$ is not normalizable — it is not in the Hilbert space $\mathcal{H}$.

We normalized $|p\rangle$ so that

$$\int dp\, |p\rangle\langle p| = \mathbb{1}, \qquad \langle p|p'\rangle = \delta(p-p')$$

But p has dimensions of momentum so the dimension of $|p\rangle$ must be the same as $1/\sqrt{p}$.

Similarly, $\int dx\, |x\rangle\langle x| = \mathbb{1}$

→ dimension of $|x\rangle$ is the same as $\frac{1}{\sqrt{x}}$.

So, the dimension of $\langle x|p\rangle$ is the same as $\frac{1}{\sqrt{px}}$, which is the same as the dimension of $\frac{1}{\sqrt{2\pi\hbar}}$.

But then $|\langle x|p\rangle|^2$ is not normalized correctly to be a probability density.

In fact, it's not normalizable:

$$\int_{-\infty}^{\infty} dx\, |\langle x|p\rangle|^2 = \infty,$$ so we might as well stick with our conventions for $|x\rangle$ and $|p\rangle$ as basis kets, and just recognize that they are not in the Hilbert space associated with the Hamiltonian $\hat{H}$.

The physical states, i.e. the states in the Hilbert space, are **wavepackets**:

$$\boxed{|\Psi(t)\rangle = \int dp\, |p\rangle e^{-i\frac{p^2}{2m\hbar}t} f(p)}$$

$$\langle\Psi(t)|\Psi(t)\rangle = 1 \rightarrow \boxed{\int dp\, |f(p)|^2 = 1}$$

N6.1 Gaussian Wavepacket

Choose $f(p) = \frac{\sqrt{d}}{(\pi\hbar^2)^{1/4}} e^{-(p-p_0)^2 d^2/2\hbar^2}$

Then at time t, $\langle\hat{p}\rangle = \langle\Psi_{p_0}(t)|\hat{p}|\Psi_{p_0}(t)\rangle$

$$= \frac{d}{(\pi\hbar^2)^{1/2}} \int_{-\infty}^{\infty} dp\, p\, e^{-(p-p_0)^2 d^2/\hbar^2}$$

$\tilde{p} = \frac{(p-p')d}{\hbar}$

$$= \frac{d}{(\pi\hbar^2)^{1/2}} \left[\left(\frac{\hbar}{d}\right)^2 \int_{-\infty}^{\infty} d\tilde{p}\, \tilde{p}\, e^{-\tilde{p}^2} + \frac{\hbar}{d} p_0 \underbrace{\int_{-\infty}^{\infty} d\tilde{p}\, e^{-\tilde{p}^2}}_{\sqrt{\pi}}\right]$$

$$\boxed{\langle\hat{p}\rangle = p_0}$$

Similarly,

$$\langle \hat{p}^2 \rangle = \frac{d}{(\pi \hbar^2)^{1/2}} \int dp \, p^2 \, e^{-(p-p')^2 d^2/\hbar^2}$$

$$\tilde{p} = \frac{(p-p')d}{\hbar}$$

$$= \frac{d}{(\pi \hbar^2)^{1/2}} \cdot \left(\frac{\hbar}{d}\right)^3 \int d\tilde{p} \left(\tilde{p}^2 + \frac{d^2}{\hbar^2} p_0^{\,2} + 2\,\tilde{p}\, \frac{p_0 d}{\hbar}\right) e^{-\tilde{p}^2}$$

← all — integrates to zero

$$= \frac{d}{(\pi \hbar^2)^{1/2}} \left[\left(\frac{\hbar}{d}\right)^3 \underbrace{\int d\tilde{p}\, \tilde{p}^2 e^{-\tilde{p}^2}}_{} + \frac{\hbar}{d} \sqrt{\pi}\, p_0^{\,2} \right]$$

$$= -\int d\tilde{p} \, \frac{d}{d\alpha} e^{-\alpha \tilde{p}^2} \Big|_{\alpha=1}$$

$$= -\frac{d}{d\alpha} \sqrt{\frac{\pi}{\alpha}} = \frac{\sqrt{\pi}}{2}$$

$$\boxed{\langle \hat{p}^2 \rangle = \frac{\hbar^2}{2d^2} + p_0^{\,2}}$$

The standard deviation of $\hat{p}$ is

$$\Delta p \equiv \sqrt{\langle (\Delta p)^2 \rangle} = \sqrt{\langle (\hat{p} - \langle \hat{p} \rangle)^2 \rangle} = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2}$$

$$= \sqrt{\left(\frac{\hbar^2}{2d^2} + p_0^{\,2}\right) - p_0^{\,2}}$$

$$\boxed{\Delta p = \frac{\hbar}{\sqrt{2}\, d}}$$

Position Space:

$$\langle x|\psi_{p_0}(t)\rangle = \int \frac{dp}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar}\, e^{-i\frac{p^2}{2m\hbar}t}$$

$$\times \frac{\sqrt{d}}{(\pi\hbar^2)^{1/4}}\, e^{-(p-p_0)^2 d^2/2\hbar^2}$$

At $t=0$: $\langle x|\psi_{p_0}(t=0)\rangle$

$$= \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar}\, \frac{\sqrt{d}}{(\pi\hbar^2)^{1/4}}\, e^{-\frac{(p-p_0)^2 d^2}{2\hbar^2}}$$

$\tilde{p} = p - p_0$

$$= \int_{-\infty}^{\infty} \frac{d\tilde{p}}{\sqrt{2\pi\hbar}}\, e^{i\tilde{p}x/\hbar}\, e^{ip_0x/\hbar}\, \frac{\sqrt{d}}{(\pi\hbar^2)^{1/4}}\, e^{-\tilde{p}^2 d^2/2\hbar^2}$$

$$= \frac{\sqrt{d}\, e^{ip_0x/\hbar}}{\hbar\sqrt{2\pi}\,\pi^{1/4}} \int_{-\infty}^{\infty} d\tilde{p}\, e^{-d^2/2\hbar^2\left(\tilde{p} - i\frac{x\hbar}{d^2}\right)^2}$$

$$\times e^{-x^2/2d^2}$$

$$= e^{ip_0x/\hbar}\, e^{-x^2/2d^2}\, \frac{\sqrt{d}}{\hbar\sqrt{2\pi}\,\pi^{1/4}} \cdot \frac{\sqrt{\pi}\cdot\hbar\sqrt{2}}{d}$$

$$\boxed{\langle x|\psi_{p_0}(t=0)\rangle = e^{ip_0x/\hbar}\, e^{-x^2/2d^2} \cdot \frac{1}{(\pi d^2)^{1/4}}}$$

Time-Evolution Operator:

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(t=0)\rangle$$

$$\hat{U}(t) = \int_{-\infty}^{\infty} dp\, |p\rangle\langle p|\, e^{-ip^2/2m\, t/\hbar}$$

$$\boxed{U(x,x';t)} \equiv \langle x|\hat{U}(t)|x'\rangle$$

$$= \int_{-\infty}^{\infty} dp\, \langle x|p\rangle\langle p|x'\rangle\, e^{-ip^2/2m\, t/\hbar}$$

$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{ip(x-x')/\hbar}\, e^{-ip^2/2m\, t/\hbar}$$

$$\boxed{= \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{im(x-x')^2/2\hbar t}}$$

$U(x,x';t)$ allows us to propagate a wavefunction in time:

$$\psi(x,t) = \int dx'\, U(x,x';t)\, \psi(x', t=0)$$

$$\psi_{p_0}(x,t) = \int dx' \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{im(x-x')^2/2\hbar t} \cdot \frac{e^{ip_0x'/\hbar}\, e^{-x'^2/2d^2}}{(\sqrt{\pi}\,d)^{1/2}}$$

$t \neq 0$; $\langle x | \psi_{p_0}(t) \rangle = \dfrac{1}{\left(\sqrt{\pi}\left(d + i\hbar t/md\right)\right)^{1/2}}$

$$\cdot \exp\left[-\frac{(x - p_0 t/m)^2}{2d^2(1 + i\hbar t/md^2)}\right]$$

$$\cdot \exp\left[-i\frac{p_0}{\hbar}\left(x - \frac{p_0 t}{2m}\right)\right]$$

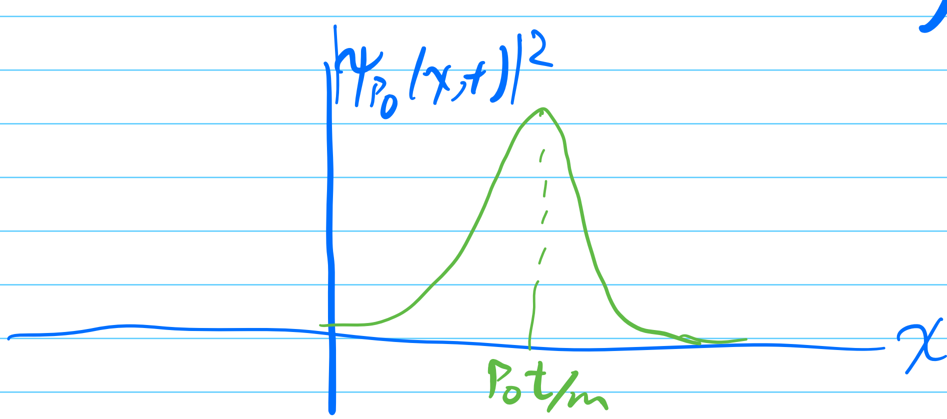
probability density :

use $\dfrac{1}{a+ib} = \dfrac{1}{a+ib}\,\dfrac{a-ib}{a-ib} = \dfrac{a-ib}{a^2+b^2}$

Let $a = 1$, $b = t\hbar/md^2$

$$|\psi_{p_0}(x,t)|^2 = \frac{1}{\sqrt{\pi}\left(d^2 + \hbar^2 t^2/m^2 d^2\right)}$$

$$\exp\left[-\frac{(x - p_0 t/m)^2}{d^2\left(1 + \hbar^2 t^2/m^2 d^4\right)}\right]$$



The probability density is a Gaussian.

The peak moves with velocity p_0/m.

The width $\Delta x = \frac{d}{\sqrt{2}}\sqrt{1+\frac{\hbar^2 t^2}{m^2 d^4}}$

By analogy w/ calculation of Δp.

grows with t

→ Approaches plane wave with momentum p_0.

Note: $$\Delta p \Delta x = \frac{\hbar}{\sqrt{2}\, d}\frac{d}{\sqrt{2}}\sqrt{1+\frac{\hbar^2 t^2}{m^2 d^4}}$$

$$= \frac{\hbar}{2}\sqrt{1+\frac{\hbar^2 t^2}{m^2 d^4}}$$

$$\Delta p \Delta x \geq \frac{\hbar}{2},$$

Gaussian at $t=0$ has minimum uncertainty.