PHYS 621 Lecture 8

Generator of spatial translations: momentum $\hat{p} = \frac{\hbar}{i}\hat{D}$

derivative operator

$$\langle x | \psi \rangle = \psi(x)$$

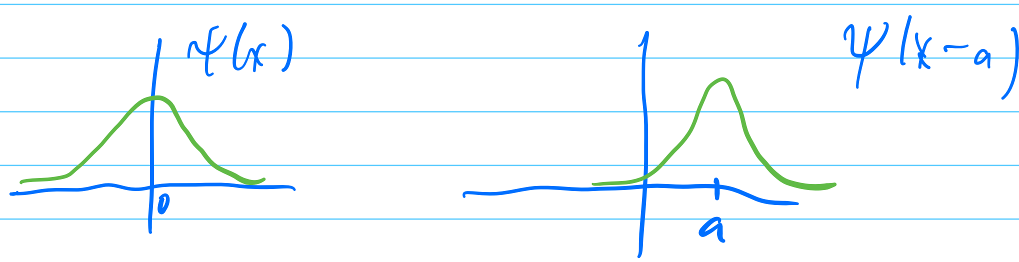
$$\langle x | \underbrace{\left(e^{-i\hat{p}a/\hbar} | \psi \rangle \right)}_{\text{Translated state}}$$

$$= \int dx' \langle x | e^{-i\hat{p}a/\hbar} | x' \rangle \langle x' | \psi \rangle$$

$$= \int dx' \langle x | x'+a \rangle \langle x' | \psi \rangle$$

$$= \int dx' \, \delta(x-(x'+a)) \, \psi(x')$$

$$= \psi(x-a)$$



Generator of time translations: Hamiltonian $\hat{H}$

$$\langle x | \psi(t) \rangle = \psi(x;t)$$

$$\underbrace{\langle x | e^{-i\hat{H}\Delta t/\hbar} | \psi(t) \rangle}_{\text{time-translated state.}} = \psi(x; t+\Delta t)$$

N5.3

Canonical Quantization

In the 1D example we have:

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$$

In the x-basis

$$\langle x'|[\hat{x},\hat{p}]|\psi\rangle = \langle x'|(\hat{x}\hat{p}-\hat{p}\hat{x})|\psi\rangle$$

$$= \int dx \langle x'|(\hat{x}\hat{p}-\hat{p}\hat{x})|x\rangle\langle x|\psi\rangle$$

$$= \int dx\, (x'-x)\langle x'|\hat{p}|x\rangle\langle x|\psi\rangle$$

integrate by parts

$$= \int dx\, (x'-x)\, i\hbar \frac{d}{dx}\delta(x-x')\,\psi(x)$$

$$= i\hbar \int dx \frac{d}{dx}\left[(x-x')\psi(x)\right]\delta(x-x')$$

$$= i\hbar \int dx \left(\underbrace{\cancel{(x-x')}\psi'(x)\delta(x-x')}_{\text{vanishes w/ } \delta\text{-fn.}} + \psi(x)\,\delta(x-x')\right)$$

$$= i\hbar\, \psi(x')$$

$$\Rightarrow \langle x'|[\hat{x},\hat{p}]|\psi\rangle = i\hbar \langle x'|\psi\rangle$$

$$\Rightarrow \boxed{[\hat{x},\hat{p}] = i\hbar}$$

In more dimensions, and with generalized coordinates q_i, with canonical momenta p_i, quantization ~ introduce observables $\hat{q}_j$, $\hat{p}_k$ such that

$$\boxed{[\hat{q}_j, \hat{p}_k] = i\hbar\, \delta_{jk}}$$

Also, $\boxed{[\hat{q}_j, \hat{q}_k] = 0}$

$$\boxed{[\hat{p}_j, \hat{p}_k] = 0}$$

N5.4 Coordinate Space vs. Momentum Space

$$\langle \psi_1 | \hat{A} | \psi_2 \rangle = \int dx \int dx' \langle \psi_1 | x \rangle \langle x | \hat{A} | x' \rangle \langle x' | \psi_2 \rangle$$

$$= \int dx \int dx' \, \psi_1(x)^* \hat{A}_{xx'} \, \psi_2(x')$$

If $\hat{A}$ is a function of $\hat{x}$ (and not $\hat{p}$), $\hat{A} = \hat{A}(\hat{x})$

$$\hat{A}_{xx'} = A(x')\, \delta(x - x')$$

$$\boxed{\langle \psi_1 | \hat{A}(x) | \psi_2 \rangle = \int dx \, \psi_1(x)^* A(x) \psi_2(x)}$$

If $\hat{A} = \hat{p}$:

$$\boxed{\langle \psi_1 | \hat{p} | \psi_2 \rangle} = \int dx \int dx' \, \psi_1(x)^* \left(-i\hbar \frac{d}{dx'} \psi(x')\right) \delta(x-x')$$

$$\boxed{= \int dx \, \psi_1(x)^* \left(-i\hbar \frac{d\psi(x)}{dx}\right)}$$

In momentum space we use momentum eigenstates

$|p\rangle$, $\quad \hat{p}|p\rangle = p|p\rangle$

$\langle p | p' \rangle = \delta(p-p')$ orthonormal

$\int dp \, |p\rangle\langle p| = \mathbb{1}$ completeness

$$\langle \psi_1 | \hat{A} | \psi_2 \rangle = \int dp \int dp' \langle \psi_1 | p \rangle \langle p | \hat{A} | p' \rangle \langle p' | \psi_2 \rangle$$

$$= \int dp \int dp' \, \psi_1(p)^* \, \hat{A}_{pp'} \, \psi_2(p')$$

Consider $\langle x | \hat{p} | p \rangle = \int dx' \langle x | \hat{p} | x' \rangle \langle x' | p \rangle$

$= p \langle x | p \rangle \qquad = \int dx' \frac{\hbar}{i} \delta(x-x') \frac{d}{dx'} \langle x' | p \rangle$

$$= \frac{\hbar}{i} \frac{d}{dx} \langle x | p \rangle$$

$$\Rightarrow \quad \frac{\hbar}{i} \frac{d}{dx} \langle x | p \rangle = p \langle x | p \rangle$$

Solution: $\boxed{\langle x | p \rangle = N e^{ipx/\hbar}}$

Completeness relation:

$$\delta(x-x') = \langle x|x'\rangle = \int dp \langle x|p\rangle\langle p|x'\rangle$$

$$= |N|^2 \int dp\, e^{ipx/\hbar} e^{-ipx'/\hbar}$$

$$= |N|^2 \cdot 2\pi\, \delta\left(\frac{x-x'}{\hbar}\right)$$

$$= |N|^2 \cdot 2\pi\hbar\, \delta(x-x')$$

$$\Rightarrow |N|^2 = 1/2\pi\hbar$$

Choose $N = \frac{1}{2\pi\hbar}$ $\rightarrow$ $\boxed{\langle x|p\rangle = \frac{1}{2\pi\hbar} e^{ipx/\hbar}}$

Aside: $\int dy\, \delta(y-y_0)\, f(y) = f(y_0)$

change of variables $\Rightarrow$ $= \int dx \left|\frac{dy}{dx}\right| \delta\left(y(x)-y_0\right) f(y(x))$

$$= f(y(x))\Big|_{x:\, y(x)=y_0}$$

$$\rightarrow \boxed{\delta(y(x)-y_0) = \frac{1}{\left|\frac{dy}{dx}\right|}\delta(x-x_0)}$$

where $y(x_0)=y_0$

$$\delta\left(\frac{x-x'}{\hbar}\right) = \hbar\, \delta(x-x')$$

Now $\langle x|\psi\rangle = \int dp \langle x|p\rangle\langle p|\psi\rangle$

$$\boxed{\psi(x) = \int \frac{dp}{\sqrt{2\pi\hbar}} e^{ipx}\, \psi(p)}$$

- $\psi(x)$ is the Fourier transform of $\psi(p)$ (normalized w/ $1/\sqrt{2\pi\hbar}$)

$\langle p|\psi\rangle = \int dx \langle p|x\rangle\langle x|\psi\rangle$

$$\boxed{\psi(p) = \int \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx}\, \psi(x)}$$

- Inverse Fourier transform