PHYS 621 Lecture Notes 7

Entanglement

Consider two systems A and B with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.

total Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

If $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$,

the state $|\psi_A\rangle \otimes |\psi_B\rangle \in \mathcal{H}_{AB}$.

But general states do not factorize this way:

$$\boxed{|\psi\rangle = \sum_{a,b} |\psi_a\rangle \otimes |\psi_b\rangle\, c_{ab}}, \quad a,b \in \{\uparrow, \downarrow\}$$

For example, consider a state of two spin-1/2 particles, in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\Big)$$

$$\equiv \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big)$$

Since the states $|\uparrow\rangle$ and $|\downarrow\rangle$ are orthogonal, so are the the tensor product states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$.

Also, $\langle\uparrow\downarrow|\uparrow\downarrow\rangle = \langle\uparrow|\uparrow\rangle \cdot \langle\downarrow|\downarrow\rangle = 1$

$\langle\downarrow\uparrow|\downarrow\uparrow\rangle = \langle\downarrow|\downarrow\rangle \cdot \langle\uparrow|\uparrow\rangle = 1.$

Hence, $\langle\psi|\psi\rangle = \frac{1}{2}\Big(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\Big)\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big)$

$= \frac{1}{2}\Big(\langle\uparrow\downarrow|\uparrow\downarrow\rangle + \langle\downarrow\uparrow|\downarrow\uparrow\rangle\Big) = 1.$

$$\text{prob}(|\uparrow\downarrow\rangle) = \left| \langle \uparrow\downarrow | \frac{(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)}{\sqrt{2}} \right|^2$$

$$= \frac{1}{2} \langle \uparrow\downarrow | \uparrow\downarrow \rangle = \frac{1}{2}.$$

Similarly, prob $(|\downarrow\uparrow\rangle) = 1/2$.

$\to \exists$ 50% prob of finding particle A w/ spin $\uparrow$ and particle B w/ spin $\downarrow$.

50 prob. of particle B w/ spin $\uparrow$ and particle A w/ spin $\downarrow$.

The density matrix in the state $|\psi\rangle$ is

$$\hat{\rho}_{AB} = \frac{1}{2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)$$

Average total z-component of spin:

$$\boxed{\langle \hat{S}_z^A + \hat{S}_z^B \rangle} = \operatorname{Tr} \hat{\rho}_{AB} \left(\hat{S}_z^A \otimes \hat{\mathbb{1}} + \mathbb{1} \otimes \hat{S}_z^B\right)$$

$$= \langle\uparrow\uparrow| \hat{\rho}_{AB} (\hat{S}_z^A \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_z^B) |\uparrow\uparrow\rangle$$

$$+ \langle\uparrow\downarrow| \hat{\rho}_{AB} (\hat{S}_z^A \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_z^B |\uparrow\downarrow\rangle$$

$$+ \langle\downarrow\uparrow| \hat{\rho}_{AB} (\hat{S}_z^A \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_z^B) |\downarrow\uparrow\rangle$$

$$+ \langle\downarrow\downarrow| \hat{\rho}_{AB} (\hat{S}_z^A \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_z^B) |\downarrow\downarrow\rangle$$

$$= 0 + \left(\tfrac{1}{2} - \tfrac{1}{2}\right)\hbar + \left(\tfrac{1}{2} - \tfrac{1}{2}\right)\hbar + 0 \; \boxed{= 0}$$

Suppose we make a measurement on system A only, without knowledge of system B.

Define $\boxed{\hat{\rho}_A = \mathrm{Tr}_B\, \hat{\rho}_{AB}}$

sum over all basis states in $\mathcal{H}_B$ only.

$$= {}_B\langle\uparrow|\psi\rangle\langle\psi|\uparrow\rangle_B + {}_B\langle\downarrow|\psi\rangle\langle\psi|\downarrow\rangle_B$$

$$= \frac{1}{2}\left(|\downarrow\rangle_A\, {}_A\langle\downarrow| + |\uparrow\rangle_A\, {}_A\langle\uparrow|\right)$$

$$\boxed{= \frac{1}{2}\hat{\mathbb{1}}_A}$$

If we are only interested in system A's observables, having already summed over system B's Hilbert space, for example

$$\langle \hat{S}_{Az}\rangle = \mathrm{Tr}_{AB}\, \hat{\rho}_{AB}\left(\hat{S}_{Az}\otimes \hat{\mathbb{1}}_B\right)$$

$$= \mathrm{Tr}_A\, \hat{\rho}_A \hat{S}_{Az}$$

$$= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}\right)\hbar = 0.$$

Forgetting about system B, the density matrix for system A corresponds to a mixed state of 50% spin-up particles and 50% spin-down particles:

$$\hat{\rho}_A = P_\uparrow |\uparrow\rangle_A \langle\uparrow| + P_\downarrow |\downarrow\rangle_A \langle\downarrow|$$

Such a state is called maximally entangled.

Quantum Computation - a brief digression.

Classical computations act on one n-bit number at a time.

A bit is a 0 or 1

(a two state system like $|\uparrow\rangle$ or $|\downarrow\rangle$).

To add 1 to the binary number 10, we have $10 + 01 = 11$.

In the quantum analogy we would have an algorithm to time-evolve the state $|10\rangle$ by a unitary time-evolution operator corresponding to addition by 1,

$$|10\rangle \longrightarrow \hat{U}|10\rangle = |11\rangle.$$

Making a measurement on the final state would reveal the outcome of the addition operation.

But in QM, entangled states allow for the possibility of acting on multiple initial configurations at once.

What does "addition by 1" do to the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$?

$$|\psi\rangle \rightarrow \hat{U}|\psi\rangle = \frac{1}{\sqrt{2}}\left(|11\rangle + |10\rangle\right).$$

We can then attempt to devise a mechanism for reading out whichever case we choose.

The state encodes both cases.

Classical computation involves bits.

Quantum computation involves q-bits.

With clever algorithms, quantum computations can be significantly faster than classical computations.

Realistic quantum computing algorithms produce results to complicated problems with high probability.

Position and Momentum

Classical phase space: x, p

Quantum phase space: $\hat{x}, \hat{p}$.

$$\hat{x}|x\rangle = x|x\rangle \qquad x\text{-basis}$$

$$\hat{p}|p\rangle = \underset{\text{eigenvalue}}{p}\,\underset{\text{eigenvector}}{|p\rangle} \qquad p\text{-basis}$$

Momentum is related to translations in space. By Noether's theorem, momentum is conserved because of spatial translation invariance.

Infinitesimal translation operator: $\hat{T}(dx)$

$$\hat{T}(dx)|x\rangle = |x+dx\rangle$$

$\hat{T}(dx)$ is unitary:

$$\langle x|\underbrace{\hat{T}(dx)^\dagger\,\hat{T}(dx)}_{\hat{1}}|x'\rangle = \langle x|x'\rangle = \delta(x-x')$$

$$\Rightarrow \boxed{\hat{T}(dx)^\dagger\,\hat{T}(dx) = \hat{1}.}$$

Translating by dx' and then dx should be the same as translating by $dx+dx'$:

$$\boxed{\hat{T}(dx)\,\hat{T}(dx') = \hat{T}(dx+dx')}$$

Inverse operator: $\boxed{\hat{T}^{-1}(dx) = \hat{T}(-dx)}$

Translation by 0: $\boxed{\hat{T}(dx \to 0) \to \hat{\mathbb{1}}}$

For infinitesimal dx, write

$$\hat{T}(dx) = \hat{\mathbb{1}} + i\,dx\,\hat{K} \quad \text{for some operator } \hat{K}.$$

$$\left.\begin{aligned} \text{Then } \hat{T}^{\dagger}(dx) &= \hat{T}^{-1}(dx) = \hat{T}(-dx) \\ &= \hat{\mathbb{1}} - i\,dx\,\hat{K} \\ &= \hat{\mathbb{1}} - i\,dx\,\hat{K}^{\dagger} \end{aligned}\right\} \quad \boxed{\hat{K} = \hat{K}^{\dagger}}$$

Consider the Taylor expansion of a wavefunction $\langle x|\psi\rangle$.

$$\psi(x - dx) = \psi(x) - dx \cdot \frac{d\psi(x)}{dx} + \cdots$$

In the x-representation, we identify

$$\langle x|\hat{T}(dx) = (\hat{T}(dx)^{\dagger}|x\rangle)^{\dagger} = (\hat{T}(-dx)|x\rangle)^{\dagger}$$

$$\langle x|\hat{T}(dx)|\psi\rangle = \langle x - dx|\psi\rangle = \psi(x - dx)$$

$$= \langle x|(\hat{\mathbb{1}} + i\,dx\,\hat{K})|\psi\rangle = \psi(x) + i\,dx\,\langle x|\hat{K}|\psi\rangle$$

$$= \psi(x) + i\,dx \int dx'\,\langle x|\hat{K}|x'\rangle\langle x'|\psi\rangle$$

$$= \psi(x) + i\,dx \int dx'\,\hat{K}_{xx'}\,\psi(x')$$

$$= \psi(x) - dx\,\frac{d\psi}{dx}$$

$$\Rightarrow \hat{K}_{xx'} = +i\,\delta(x-x')\frac{d}{dx'}$$

$$\boxed{\hat{K} = i\,\hat{D}}$$

$\langle x|\hat{D}|x'\rangle = \delta(x-x')\frac{d}{dx'}$

derivative operator

$$\hat{X}\,\hat{T}(dx)\,|x\rangle = \hat{X}\,|x+dx\rangle = (x+dx)\,|x+dx\rangle$$

$$\hat{T}(dx)\,\hat{X}\,|x\rangle = \hat{T}(dx)\,x\,|x\rangle = x\,|x+dx\rangle$$

$$\Rightarrow [\hat{X}, \hat{T}(dx)] = \hat{X}\hat{T}(dx) - \hat{T}(dx)\hat{X} = dx\,\hat{\mathbb{1}}$$

$$[\hat{X}, \hat{\mathbb{1}} + i\,dx\,\hat{K}] = i\,dx\,[\hat{X}, \hat{K}] = dx\,\hat{\mathbb{1}}$$

$$\Rightarrow \boxed{[\hat{X}, \hat{K}]} = -i\,\hat{\mathbb{1}} + O(\cancel{d}x)$$

Define $\boxed{\hat{P} = \hbar\,\hat{K} = -i\hbar\,\hat{D}}$, $\langle x|\hat{P}|x'\rangle = -i\hbar\,\delta(x-x')\frac{d}{dx'}$

$= -i\hbar\frac{d}{dx}\delta(x-x')$

Consider time-evolution of the translated state:

$$i\frac{d}{dt}\hat{T}(dx)|\psi(t)\rangle = \hat{T}(dx)\hat{H}\,|\psi(t)\rangle$$

If $[\hat{H}, \hat{T}(dx)] = 0$, i.e. $[\hat{H}, \hat{P}] = 0$,

Then $i\frac{d}{dt}\hat{T}(dx)|\psi(t)\rangle = \hat{T}(dx)\hat{H}\,|\psi(t)\rangle = \hat{H}\,\hat{T}(dx)|\psi(t)\rangle$

$$i\frac{d}{dt}\langle x|\hat{T}(dx)|\psi(t)\rangle = \int dx'\,\langle x|\hat{H}|x'\rangle\langle x'|\hat{T}(dx)|\psi(t)\rangle$$

$$\rightarrow \quad \boxed{i\frac{\partial}{\partial t}\psi(x+dx,t) = \int dx'\, H_{x,x'}\, \psi(x'+dx, t)}$$

i.e. the translated wave function satisfies the <u>same</u> Schrödinger eqn

This is what we mean by spatial translation invariance.

We used $[\hat{H}, \hat{P}] = 0$ for this, so

$$\boxed{[\hat{H}, \hat{P}] = 0 \sim \text{spatial translation invariance}}$$

<u>Conservation Law associated w/ translation invariance</u>

Consider any states $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$ satisfying the Schrödinger Eq:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$$

$$-i\hbar\frac{d}{dt}\langle\psi(t)| = \langle\psi(t)|\hat{H}^{\dagger} = \langle\psi(t)|\hat{H}$$

Consider the matrix element $\langle\psi_1(t)|\hat{P}|\psi_2(t)\rangle$.

$$\frac{d}{dt}\langle\psi_1|\hat{P}|\psi_2\rangle = \left(\frac{d}{dt}\langle\psi_1|\right)\hat{P}|\psi_2\rangle + \langle\psi_1|\hat{P}\left(\frac{d}{dt}|\psi_2\rangle\right).$$

$$= \frac{1}{i\hbar}\left(-\langle\psi_1|\hat{H}\hat{P}|\psi_2\rangle + \langle\psi_1|\hat{P}\hat{H}|\psi_2\rangle\right)$$

$$= -\frac{1}{i\hbar}\langle\psi_1|[\hat{H},\hat{P}]|\psi_2\rangle \quad = 0$$

$$\rightarrow \boxed{\frac{d}{dt}\langle\psi_1|\hat{P}|\psi_2\rangle = 0}$$ for any states $|\psi_1\rangle, |\psi_2\rangle$.

★ Hence, the observable $\hat{P}$ is conserved.

★ The observable that is conserved by virtue of spatial translation invariance is called the <u>momentum</u>.

$$\rightarrow \boxed{\hat{P} = -i\hbar\hat{D}}$$ Momentum operator.

$$\boxed{[\hat{X},\hat{P}] = i\hbar}$$

The constant $\hbar$ agrees w/ de Broglie relation for a plane wave:

Suppose $\psi(x,t) = e^{-i(\omega t - \overset{\text{wave vector}}{k} x)}$

$$-i\hbar\frac{\partial\psi}{\partial x} = \hbar k\,\psi(x,t)$$ — eigenstate of $\hat{P}$ w/ eigenvalue $\boxed{p = \hbar k}$

Macroscopic Translations

$$\psi(x-a) = \psi(x) - a\frac{d\psi}{dx} + \frac{a^2}{2}\frac{d^2}{dx^2}\psi(x)$$

$$+ \cdots = \sum_{n=0}^{\infty}\frac{(-a)^n}{n!}\frac{d^n}{dx^n}\psi(x)$$

$$\equiv \exp\left(-a\frac{d}{dx}\right)\psi(x)$$

exponential of differential operator — defined by Taylor expansion.

$$= \exp\left[-\frac{ia}{\hbar}\left(\frac{\hbar}{i}\frac{d}{dx}\right)\right]\psi(x)$$

$$\hat{T}(a)|x\rangle = |x+a\rangle$$

$$\boxed{\hat{T}(a) = \exp\left(-a\hat{D}\right) = \exp\left(-ia\frac{\hat{P}}{\hbar}\right)}$$

You can also build up a macroscopic translation from infinitesimal ones.

Let $dx = \frac{a}{N}$

$$\hat{T}(a) = \left[\hat{T}(dx)\right]^N = \lim_{N\to\infty}\left(\mathbb{1} - idx\frac{\hat{P}}{\hbar}\right)^N = \exp\left(-ia\frac{\hat{P}}{\hbar}\right)$$