PHYS 621 Lecture Notes 6

N 4.2 General Two-State Systems

Suppose a Hamiltonian $\hat{H}$ acts on a 2-dimensional Hilbert space $\mathcal{H}$ with complete orthonormal basis $|\phi_1\rangle, |\phi_2\rangle$.

Write $|\psi(t)\rangle = c_1(t)|\phi_1\rangle + c_2(t)|\phi_2\rangle$

Represent $|\phi_1\rangle$ by $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|\phi_2\rangle$ by $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|\psi(t)\rangle = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$i\hbar \frac{\partial}{\partial t}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

Hermitian matrix $H_{11}, H_{22} \in \mathbb{R}$

Eigenstates of $\hat{H}$:

$$i\hbar \frac{\partial}{\partial t}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = E\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

eigenvalue eigenvector

Solution: $\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = e^{-iEt/\hbar} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

constant

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = E \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Nontrivial solution requires

$$\det \begin{pmatrix} H_{11}-E & H_{12} \\ H_{12}^* & H_{22}-E \end{pmatrix} = 0$$

$$(H_{11}-E)(H_{22}-E) - |H_{12}|^2 = 0$$

$$E^2 - E(H_{11}+H_{22}) + (H_{11}H_{22} - |H_{12}|^2) = 0$$

solution for eigenvalues:

$$\boxed{E_\pm = \frac{(H_{11}+H_{22}) \pm \sqrt{(H_{11}-H_{22})^2 + 4|H_{12}|^2}}{2}}$$

Eigenvectors satisfy

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix} \begin{pmatrix} c_1^\pm \\ c_2^\pm \end{pmatrix} = E_\pm \begin{pmatrix} c_1^\pm \\ c_2^\pm \end{pmatrix}$$

$|\psi_\pm\rangle$

$$H_{11} c_1^{\pm} + H_{12} c_2^{\pm} = E_{\pm} c_1^{\pm}$$

$$\rightarrow c_2^{\pm} = \left(\frac{E_{\pm} - H_{11}}{H_{12}} \right) c_1^{\pm}$$

Normalization: $|c_1^{\pm}|^2 + |c_2^{\pm}|^2 = 1$

$$\rightarrow \begin{pmatrix} c_1^{\pm} \\ c_2^{\pm} \end{pmatrix} = \frac{1}{\sqrt{1 + \left| \frac{E_{\pm} - H_{11}}{H_{12}} \right|^2}} \begin{pmatrix} 1 \\ \frac{E_{\pm} - H_{11}}{H_{12}} \end{pmatrix}$$

can multiply by a phase $e^{i\theta}$

$$\langle \psi_- | \psi_+ \rangle = \frac{1}{\sqrt{1 + \left| \frac{E_- - H_{11}}{H_{12}} \right|^2}} \frac{1}{\sqrt{1 + \left| \frac{E_+ - H_{11}}{H_{12}} \right|^2}}$$

$$\times \left(1, \frac{E_- - H_{11}}{H_{12}^*} \right) \begin{pmatrix} 1 \\ \frac{E_+ - H_{11}}{H_{12}} \end{pmatrix}$$

But $\left(1, \frac{E_- - H_{11}}{H_{12}^*} \right) \begin{pmatrix} 1 \\ \frac{E_+ - H_{11}}{H_{12}} \end{pmatrix} = 1 + \frac{(H_{22} - H_{11})^2 - \left((H_{11} - H_{22})^2 + 4|H_{12}|^2 \right)}{4|H_{12}|^2}$

$$= 0$$

So, $\langle \psi_- | \psi_+ \rangle = 0$, and $\langle \psi_- | \psi_- \rangle = \langle \psi_+ | \psi_+ \rangle = 1$.

General time-dependent solution to Schrödinger Eq:

$$\boxed{|\psi(t)\rangle = c_- e^{-iE_- t/\hbar} |\psi_-\rangle + c_+ e^{iE_+ t/\hbar} |\psi_+\rangle}$$

$c_- = \langle \psi_- | \psi(t) \rangle$, $c_+ = \langle \psi_+ | \psi(t) \rangle$

Oscillations of States

Suppose the Hamiltonian is of the form $\hat{H} = \begin{pmatrix} E-\Delta & V \\ V & E+\Delta \end{pmatrix}$

special case: $H_{12} = H_{21} = V \in \mathbb{R}$.

Then $c_1^{\pm}, c_2^{\pm} \in \mathbb{R}$, $(c_1^{\pm})^2 + (c_2^{\pm})^2 = 1$

→ can find θ such that

$$\begin{pmatrix} c_1^- \\ c_2^- \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \begin{pmatrix} c_1^+ \\ c_2^+ \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Equivalently,

$$|\psi_-\rangle = \cos\theta\,\underset{\begin{pmatrix}1\\0\end{pmatrix}}{|\phi_1\rangle} + \sin\theta\,\underset{\begin{pmatrix}0\\1\end{pmatrix}}{|\phi_2\rangle}$$

$$|\psi_+\rangle = -\sin\theta\,|\phi_1\rangle + \cos\theta\,|\phi_2\rangle$$

From our solutions above

$$\cos\theta = \frac{1}{\sqrt{1+\left|\frac{E_- - H_{11}}{H_{12}}\right|^2}} = \frac{1}{\sqrt{1+\frac{\left((H_{22}-H_{11})-\sqrt{(H_{22}-H_{11})^2+4|H_{12}|^2}\right)^2}{2\,|H_{12}|^2}}}$$

$$= \frac{V}{\sqrt{V^2+\left(\Delta-\sqrt{\Delta^2+V^2}\right)^2}}$$

Suppose the state at $t=0$ is

$$|\psi(t=0)\rangle = \underset{\text{basis ket}}{|\phi_1\rangle} = \cos\theta\,|\psi_-\rangle - \sin\theta\,|\psi_+\rangle$$

Hamiltonian eigenstates.

$$c_\pm = \langle\psi_\pm|\psi(t=0)\rangle$$

$$\rightarrow \quad c_- = \cos\theta \quad , \quad c_+ = -\sin\theta$$

→ Solution to Schrödinger Eq.:

$$|\psi(t)\rangle = c_-\, e^{-iE_-t/\hbar}|\psi_-\rangle + c_+ e^{-iE_+t/\hbar}|\psi_+\rangle$$

$$= \cos\theta\, e^{-iE_-t/\hbar}\left(\cos\theta\,|\phi_1\rangle + \sin\theta\,|\phi_2\rangle\right)$$

$$- \sin\theta\, e^{-iE_+t/\hbar}\left(-\sin\theta\,|\phi_1\rangle + \cos\theta\,|\phi_2\rangle\right)$$

$$= \left(\cos^2\theta\, e^{-iE_-t/\hbar} + \sin^2\theta\, e^{-iE_+t/\hbar}\right)|\phi_1\rangle$$

$$+ \left(\cos\theta\sin\theta\, e^{-iE_-t/\hbar} - \cos\theta\sin\theta\, e^{-iE_+t/\hbar}\right)|\phi_2\rangle$$

What is the probability of finding the state in the basis state $|\phi_2\rangle$?

$$P_2(t) = |\langle\phi_2|\psi(t)\rangle|^2 \qquad \text{Born rule}$$

$$P_2(t) = (\cos\theta \sin\theta)^2 \left| e^{-iE_- t/\hbar} - e^{-iE_+ t/\hbar} \right|^2$$

$$= \left(\frac{\sin 2\theta}{2}\right)^2 \left(1 + 1 - e^{-i(E_+ - E_-)t/\hbar} - e^{+i(E_+ - E_-)t/\hbar}\right)$$

$$= \frac{\sin^2 2\theta}{2} \left(1 - \cos\left((E_+ - E_-)t/\hbar\right)\right)$$

With our Hamiltonian

$$\boxed{P_2(t) = \frac{V^2}{2(\Delta^2 + V^2)} \left(1 - \cos \Omega t\right)}$$

where $\boxed{\Omega = \frac{E_+ - E_-}{\hbar} = \frac{2\sqrt{\Delta^2 + V^2}}{\hbar}}$

$\Rightarrow$ The probability of finding the state in one of the basis states oscillates with angular frequency Ω.

This is a good approximation to how neutrinos oscillate:

The flavor eigenstates (electron, muon, tau) $\sim (|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle)$

Hamiltonian eigenstates $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ are linear combinations of the flavor states.

$\rightarrow$ The flavor states oscillate into one another over time.

Note: If $V = 0$ then $P_2(t) = 0$ $\rightarrow$ No oscillations

Unitary time-evolution operator

Write $|\psi(t)\rangle = \hat{U}(t)\,|\psi(t=0)\rangle$

If $|\psi(t=0)\rangle = \begin{pmatrix} b \\ a \end{pmatrix}$ then

$$U(t) = e^{-i\bar{E}t/\hbar}\begin{pmatrix} \cos^2\theta\, e^{i\Omega t/2} + \sin^2\theta\, e^{-i\Omega t/2} & i\sin 2\theta \sin\frac{\Omega t}{2} \\ i\sin 2\theta \sin\frac{\Omega t}{2} & \sin^2\theta\, e^{i\Omega t/2} + \cos^2\theta\, e^{-i\Omega t/2} \end{pmatrix}$$

Exercise.