PHYS 621 Lecture Notes 3

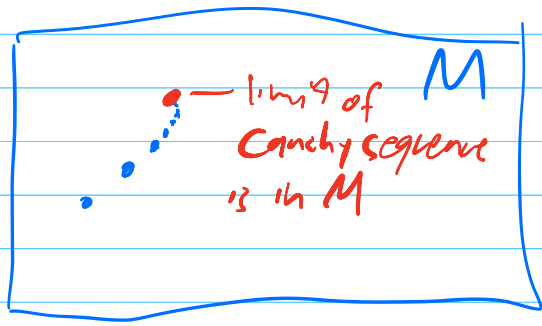
N2.1 ## Infinite Dimensional Hilbert Spaces

Metric Space — a vector space with a metric (distance function) $d(x,y)$ defined on it.

Triangle inequality: $d(x,z) \le \boxed{d(x,y) + d(y,z)}$

Cauchy space = complete metric space:

Cauchy sequences of pts in the space M have limits in the space M.



Cauchy sequence: Sequence of pts s.t. $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$

★ **Hilbert space** = complex vector space endowed with a scalar product $\langle x | y \rangle$ such that $d(x,y) = \|x-y\| = \sqrt{\langle x-y | x-y \rangle}$ and which is a Cauchy space with respect to this metric.

N2.2 **Functions** – functions can be added together or multiplied by scalars to give other functions → they form a vector space, but an infinite-dimensional one.

Operators act on functions and give other functions.

Ex: $\frac{d}{dx} : f(x) \to f'(x)$

Q: How should we define a scalar product $\langle f(x) | g(x) \rangle$?

First, consider a function over a finite discrete set of pts $\{x_i\}$, $i \in \{1, \dots, n\}$

Define an orthonormal basis $|x_i\rangle$.

$$\langle x_i | x_j \rangle = \delta_{ij}$$

complete: $\sum_{i=1}^{n} |x_i\rangle\langle x_i| = \hat{\mathbb{1}}$

$$\to |f\rangle = \sum_{i=1}^{n} f(x_i)\,|x_i\rangle$$

$$= \sum_{i=1}^{n} |x_i\rangle\langle x_i | f\rangle$$

$$\to \boxed{f(x_i) = \langle x_i | f \rangle}$$

→ Scalar product between $|f\rangle$ and $|g\rangle$

is

$$\boxed{\langle f|g\rangle} = \sum_{i=1}^{n} \langle f|x_i\rangle\langle x_i|g\rangle$$

$$\boxed{= \sum_{i=1}^{n} f(x_i)^* g(x_i)}$$

Norm of $|f\rangle$:

$$\boxed{\|f\|^2 = \langle f|f\rangle = \sum_{i=1}^{n} |f(x_i)|^2}$$

N2.3 Limit as $n \to \infty$:

Let $\Delta x_i = x_{i+1} - x_i = \epsilon$

$\Delta x_i \to dx$ as $\epsilon \to 0$.

$$\sum_{i=1}^{n} f(x_i) = \int_a^b dx\, f(x)$$

$\Delta x_i = \epsilon = \frac{b-a}{n}$ as $n \to \infty$

- Riemann sum.

Square integrable functions $L^2(x)$:

$$\int |f(x)|^2 dx < \infty$$

→ $\boxed{\langle f|g\rangle = \int f^*(x) g(x)\, dx}$ well defined

Distributions – Can be integrated against test functions with compact support (i.e. $\mathbb{C}^\infty$ functions vanish outside a compact domain).

infinitely differentiable

locally integrable functions

Example: $f(x) \in \mathcal{L}^1_{loc}(A)$, $A \in X$

$$\int_A f(x)\,dx < \infty$$

Then for test function $g(x)$,

$$\langle f | g \rangle = \int_A f^*(x) g(x)\,dx$$

distribution associated with $f(x)$

function

Example: Dirac delta function:

$$\langle \delta_{x_0} | g \rangle = \text{“}\int_A \delta(x-x_0)\, g(x)\,dx\text{”}$$

$x_0 \in A$

$$= g(x_0)$$

Distribution, Not associated with a function

– Not a Riemann integral, but still an integral

Distributions can be multiplied by C^∞ functions to give new distributions:

$$\langle f(x)\,\delta_{x_0} \mid g \rangle = f^*(x_0)\,g(x_0)$$

There is no function associated with δ_{x_0}

- it is an <u>irregular</u> distribution.
  - still well defined, just not associated with a function.

<u>Example</u>: Heaviside distribution

$$\langle H_{x_0} \mid g \rangle = \int_{x_0}^{\infty} g(x)\,dx$$

Heaviside function associated with the distribution H_{x_0}:

$$H_{x_0}(x) = \begin{cases} 1 & \text{if } x > x_0 \\ 0 & \text{if } x \leq x_0 \end{cases}$$

- Not differentiable as a function at x_0
- Differentiable as a distribution.

Derivatives of distributions - defined as in integration by parts:

$$\left\langle \frac{\partial}{\partial x} \delta_{x_0} \middle| g \right\rangle = - \left\langle \delta_{x_0} \middle| \frac{dg}{dx} \right\rangle$$

$$= -g'(x_0)$$

Example: $\left\langle \frac{dH_{x_0}}{dx} \middle| g \right\rangle = - \left\langle H_{x_0} \middle| \frac{dg}{dx} \right\rangle$

$$= - \int_{x_0}^{\infty} g'(x)\,dx$$

$$= g(x_0) \quad \text{for test fn. } g(x).$$

But this is the same as

$$\langle \delta_{x_0} | g \rangle = g(x_0)$$

$\rightarrow$ Identify $\boxed{H'_{x_0} = \delta_{x_0}}$

Function spaces

we can now define the continuum analogy to

$$\langle x_i | x_j \rangle = \delta_{ij} :$$

$$\boxed{\langle x | y \rangle = \delta(x-y)} :$$

$$\int_a^b \delta(x-y)\, dy = 1 \quad , f \quad a < x < b$$

$$\sum_i |x_i\rangle\langle x_i| = \mathbb{1} \rightsquigarrow \int dx\, |x\rangle\langle x| = \mathbb{1}$$

$$\langle x|f\rangle = \int dx' \langle x|x'\rangle\langle x'|f\rangle$$

$$= \int dx'\, \delta(x-x')\, f(x')$$

$$\boxed{\langle x|f\rangle = f(x)}$$

Representations of the δ fn.

$$\delta(x-y) \sim \lim_{\Delta \to 0} \frac{1}{\sqrt{\pi \Delta^2}} e^{-(x-y)^2/\Delta^2}$$

$$\delta(x-y) = \delta(y-x) \qquad \underline{\text{even}}$$

$$\delta'(x-y) = -\delta'(y-x) = \delta(x-y)\frac{d}{dy}$$

N2.6 **Operators:** Derivative operator $\hat{D}$

$$\boxed{\hat{D}|f\rangle = \left|\frac{df}{dx}\right\rangle}$$

$$\boxed{\langle x|\hat{D}|f\rangle = \langle x|\frac{df}{dx}\rangle = f'(x)}$$

Using $\int dx' \, |x'\rangle\langle x'| = \mathbb{1}$:

$$\int dx' \langle x|\hat{D}|x'\rangle\langle x'|f\rangle = f'(x)$$

$$\rightarrow \boxed{D_{xx'} \equiv \langle x|\hat{D}|x'\rangle = \delta(x-x')\frac{d}{dx'}}$$

N2.7 **Hermitian Operators and Eigenvalue problems**

$\hat{A}: H \rightarrow H$ a linear operator

Hilbert Space.

Adjoint operator $\hat{A}^{+}$ defined s.t.

$$\boxed{\langle \hat{A}^{+} f | g\rangle = \langle f|\hat{A}\, g\rangle}$$

Hermitian = Self-adjoint : $\hat{A}^{+} = \hat{A}$

$$\rightarrow \langle \hat{A} f|g\rangle = \langle f|\hat{A} g\rangle$$

Assume $f, g \in H$.

Theorem: Hermitian $\hat{A}$ is bounded.

Theorem: Hermitian $\hat{A}$ has $\langle \hat{A} f | f \rangle \in \mathbb{R}$ (real)

Theorem: For Hermitian $\hat{A}$, different eigenvalues → orthogonal eigenvectors

Theorem (Hilbert-Schmidt): Consider the eigenvalue problem

$$(\hat{A} - \lambda \hat{\mathbb{1}})|f\rangle = 0.$$

For Hermitian $\hat{A}$, the eigenvectors are a complete basis:

$$\hat{A}|f\rangle = \sum_{i=1}^{\infty} \underset{\text{eigenvalue}}{\lambda_i} \langle f_i | f \rangle \underset{\text{eigenvector}}{|f_i\rangle}$$

N2.8 Q: Is the derivative operator Hermitian?

$$D_{xx'} = \langle x | \hat{D} | x' \rangle = \delta(x-x') \frac{d}{dx'}$$

$$\langle f | \hat{D} g \rangle = \int dx \int dx' \langle f | x \rangle \langle x | \hat{D} | x' \rangle \langle x' | g \rangle$$

$$= \int dx \int dx' f(x)^* \delta(x-x') \frac{d}{dx'} g(x')$$

$$= \int dx\, f(x)^* \frac{d}{dx} g(x)$$

$$= -\int dx \left(\frac{d}{dx} f(x)\right)^* g(x)$$

$$= -\langle g | \hat{D} f \rangle^*$$

$$= - \langle \hat{D} f | g \rangle$$

→ $\hat{D}$ is not a Hermitian operator.

But $-i\hat{D}$ is Hermitian:

$$\langle f | -i\hat{D} g\rangle = -i\langle f | \hat{D} g\rangle$$

$$= -i\langle g | \hat{D} f\rangle^* \quad \text{(from before)}$$

$$= \langle g | i\hat{D} f\rangle^* \quad (\text{using } i^* = -i)$$

$$= \langle -i\hat{D} f | g\rangle \quad (\text{using } i^* = -i)$$

$$\rightarrow (-i\hat{D})^\dagger = (-i\hat{D})$$

Caveat: when we integrated by parts we ignored boundary terms. What if the function $f(x)$ or $g(x)$ has finite domain, say $f(x)=0$ if $x<a$ or $x>b$?

Then $\langle f | -i\hat{D} g\rangle = \langle -i\hat{D} f | g\rangle + i\, f^*(x) g(x) \Big|_a^b$

surface terms from integration by parts.

★ $\rightarrow$ $-i\hat{D}$ is Hermitian with the boundary condition $f^*(x) g(x) \Big|_a^b = 0$ under these conditions

Eigenvalue problem for $-i\hat{D}$: $\quad -i\hat{D}|k\rangle = k|k\rangle$

eigenvalue eigenvector

$$\int dx' \langle x| -i\hat{D} |x'\rangle\langle x'|k\rangle = k\langle x|k\rangle$$

$$= \int dx' \, -i\,\delta(x-x') \frac{d}{dx'} \langle x'|k\rangle$$

Write $\psi_k(x) \equiv \langle x|k\rangle$

$$\rightarrow -i\frac{d}{dx}\psi_k(x) = k\,\psi_k(x)$$

Solution: $\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$

$$\langle k'|k\rangle = \int dx\, \langle k'|x\rangle\langle x|k\rangle$$

$$= \int dx \cdot \frac{1}{2\pi} e^{-ik'x}\, e^{ikx}$$

A representation of the δ-function

$$\boxed{\langle k'|k\rangle = \delta(k-k')}$$ — Eigenvectors are orthogonal

In the $|k\rangle$ basis:

$$\boxed{\begin{aligned}\langle k'|(-i\hat{D})|k\rangle &= k\,\langle k'|k\rangle \\ &= k\,\delta(k-k')\end{aligned}}$$