PHYS 621 Lecture Notes 22

Gauge Transformations and the Aharonov-Bohm Effect

The vector potential $\vec{A}(\vec{x})$ in the presence of a magnetic field $\vec{B}(\vec{x})$ satisfies $\vec{B} = \nabla\times\vec{A}$.

This does not uniquely define $\vec{A}$. Replacing $\vec{A}(\vec{x})$ with

$$\boxed{\vec{A}'(\vec{x}) \equiv \vec{A}(\vec{x}) + \nabla\Lambda(\vec{x})}$$ for some function $\Lambda(\vec{x})$

– gauge transformation – depends on $\vec{x}$.

satisfies $$\nabla\times\vec{A}' = \nabla\times\vec{A} + \nabla\times\nabla\Lambda$$
$$= \vec{B} + \vec{0}$$

because $$(\nabla\times\nabla\Lambda)_i = \underbrace{\epsilon_{ijk}}_{\text{antisymmetric in } j\leftrightarrow k}\ \underbrace{\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}}_{\text{symmetric in } j\leftrightarrow k}\Lambda = 0$$

If $\vec{B}(\vec{x})$ is measurable but $\vec{A}(\vec{x})$ is not, then observable quantities should not depend on the specific form of $\vec{A}(\vec{x})$, i.e. the choice of $\Lambda(\vec{x})$.

The Hamiltonian for a charged particle in an external static magnetic field is

$$\hat{H} = \frac{\vec{P}_{kin}^2}{2m}$$, where the kinetic momentum $\vec{P}_{kin}$ is

$$\boxed{\vec{P}_{kin} = \vec{P} - q\vec{A}(\vec{x})}$$

q ← charge of particle

Under a gauge transformation, $\vec{P}_{kin} \to \vec{P}_{kin}' = \vec{P}_{kin} - q\nabla\Lambda(\vec{x})$

Suppose we replace $\Psi(\vec{x},t)$ with $\Psi'(\vec{x},t) = \Psi(\vec{x},t)e^{iq\Lambda(\vec{x})/\hbar}$

$$\vec{P}_{kin}'\Psi' = \left(\frac{\hbar}{i}\nabla - q\vec{A} - q(\nabla\Lambda)\right)\left(\Psi(\vec{x},t)e^{iq\Lambda(\vec{x})/\hbar}\right)$$

$$\vec{P}_{kin}'\psi' = e^{\frac{iq\Lambda(\vec{x})}{\hbar}}\left[\vec{P}_{kin}\psi - q(\nabla\Lambda)\psi + q(\nabla\Lambda)\psi\right]$$

from $\frac{\hbar}{i}\nabla e^{iq\Lambda/\hbar}$

$$= e^{\frac{iq\Lambda(\vec{x})}{\hbar}}\left(\vec{P}_{kin}\psi(\vec{x},t)\right)$$

More generally, $\left(\vec{P}_{kin}'\right)^n\psi' = e^{\frac{iq\Lambda(\vec{x})}{\hbar}}\left(\left(\vec{P}_{kin}\right)^n\psi\right)$

Hence, $\exp\left[-\frac{i\vec{P}_{kin}'^{\,2}}{2m\hbar}t\right]\psi' = e^{iq\Lambda(\vec{x})}\exp\left[-\frac{i\vec{P}_{kin}^{\,2}}{2m\hbar}t\right]\psi$.

Further, if $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$, then

$$i\hbar\frac{\partial\psi'}{\partial t} = e^{iq\Lambda(\vec{x})/\hbar}\, i\hbar\frac{\partial\psi}{\partial t}$$
$$= e^{iq\Lambda(\vec{x})/\hbar}\hat{H}\psi$$
$$= \hat{H}'\psi'$$

Hence, given a solution to the Schrödinger Eq. $\psi(\vec{x},t)$,

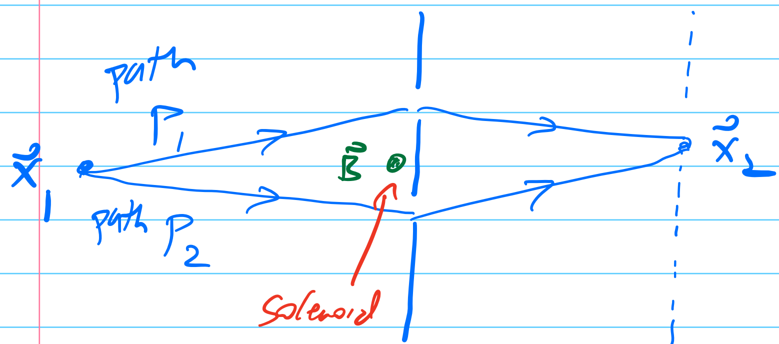
the gauge transformation $\vec{A}\to\vec{A}' = \vec{A}+\nabla\Lambda$ simply results in replacement of ψ with ψ multiplied by a position dependent phase $e^{iq\Lambda(\vec{x})/\hbar}$.

Note that this phase doesn't depend on the solution $\psi(\vec{x},t)$ that we start with.

Hence, $\langle\psi_2(t)|f(\vec{x},\vec{P}_{kin})|\psi_1(t)\rangle = \langle\psi_2'(t)|f(\vec{x},\vec{P}_{kin}')|\psi_1'(t)\rangle$

So the gauge transformation is not observable.

Now consider the double slit experiment with a small solenoid between the slits such that the classical paths of the electron are not close to the region with the nonvanishing magnetic field.



To translate a state from $\vec{x}_1$ to $\vec{x}_2$, we act with the translation operator

$$\exp\left(\frac{i}{\hbar}\int_{\substack{\text{path } P \text{ from} \\ \vec{x}_1 \text{ to } \vec{x}_2}} \hat{\vec{p}}\cdot d\vec{x}\right)$$

$$= \exp\left[\frac{i}{\hbar}\int_{\text{path } P}\left(\hat{\vec{P}}_{\text{kin}} + q\hat{\vec{A}}\right)\cdot d\vec{x}\right]$$

$$= \exp\left[\frac{i}{\hbar}\int_{\text{path } P}\hat{\vec{P}}_{\text{kin}}\cdot d\vec{x}\right] e^{i\delta}$$

$$\delta = \frac{q}{\hbar}\int_{\substack{\text{path } P \text{ from} \\ \vec{x}_1 \text{ to } \vec{x}_2}} \vec{A}\cdot d\vec{x}$$

gauge invariant - physical observable

gauge dependent

Under a gauge transformation, the spatial translation operator transforms as

$$\exp\left(\frac{i}{\hbar}\int\vec{p}\cdot d\vec{x}\right) \longrightarrow \exp\left(\frac{i}{\hbar}\int\vec{p}\cdot d\vec{x}\right)\exp\left(\frac{iq}{\hbar}\int\nabla\Lambda\cdot d\vec{x}\right)$$

In the double slit experiment we imagine a superposition of two wavefunctions ψ_1 and ψ_2 corresponding to the two paths:

$$\psi_1(\vec{x}_2,t) = \exp\left(\frac{i}{\hbar}\int_{\text{path 1}} \vec{p}_{kin}\cdot d\vec{x}\right)\psi_1(\vec{x}_1,t)\, e^{iq/\hbar \int_{P_1}\vec{A}\cdot d\vec{x}}$$

$$\psi_2(\vec{x}_2,t) = \exp\left(\frac{i}{\hbar}\int_{\text{path 2}} \vec{p}_{kin}\cdot d\vec{x}\right)\psi_1(\vec{x}_1,t)\, e^{iq/\hbar \int_{P_2}\vec{A}\cdot d\vec{x}}$$

$$\psi(\vec{x}_2,t) = N\left[\psi_1(\vec{x}_2,t) + \psi_2(\vec{x}_2,t)\right]$$

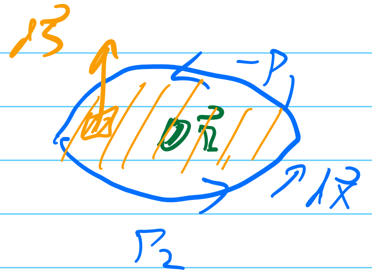
$$= N\,\psi_1(\vec{x}_2,t)\left(1 + e^{iq/\hbar \int_{P_2-P_1}\vec{A}\cdot d\vec{x}}\right)$$

$$= N\,\psi_1(\vec{x}_2,t)\left(1 + e^{\underbrace{iq/\hbar \oint \vec{A}\cdot d\vec{x}}_{\text{Aharonov-Bohm phase}}}\right)$$

☆ Even though the paths P_1 and P_2 do not pass through the region of the magnetic field, the wavefunction at $\vec{x}_2$ includes a term that depends on $\vec{B}$ (through $\vec{A}$) that will affect the interference pattern observed at the screen ($\vec{x}_2$).

Note that $\oint \vec{A}\cdot d\vec{x} = \int (\nabla\times\vec{A})\cdot d\vec{S}$

$$= \int \vec{B}\cdot d\vec{S}$$



The relative phase between the wavefunctions along the two paths has the form

$$e^{i\frac{q\Phi}{\hbar}}$$, where $\Phi = \int \vec{B}\cdot d\vec{S}$ is the flux of the magnetic field through (any) surface bounded by the loop $P_2 - P_1$

Note that, if the flux has the value

$$\Phi = n \cdot \frac{2\pi\hbar}{q} \quad \text{for } n \in \mathbb{Z},$$

the Aharonov – Bohm phase is a multiple of 2π, and hence is unobservable.

The flux quantum $\frac{2\pi\hbar}{q}$ shows up in many quantum systems where magnetic effects are important, for example in superconductors