PHYS 621 Lecture Notes 20

Spin and $\vec{L} + \vec{S}$

We introduced the concept of spin early in the semester in the context of the Stern-Gerlach experiment.

To expand on that discussion, we are now in a better position to consider the interaction of the quantum magnetic moment with a magnetic field.

Classical E&M: Define a vector potential $\vec{A}$ such that

$$\nabla \times \vec{A} = \vec{B}.$$

For a uniform $\vec{B}$, we can choose $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$

Check: $$\left(\nabla \times \left[\tfrac{1}{2}\vec{B}\times\vec{r}\right]\right)_i = \epsilon_{ijk}\,\partial_j\left(\epsilon_{klm}\tfrac{1}{2}B_l x_m\right)$$

(using $(\vec{A}\times\vec{B})_i = \epsilon_{ijk}A_jB_k$

$$= \underbrace{\epsilon_{ijk}\epsilon_{klm}}_{(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})}\tfrac{1}{2}B_l\underbrace{(\partial_j x_m)}_{\delta_{jm}}$$

$$= \tfrac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)B_l\,\delta_{jm}$$

$$= \tfrac{1}{2}\left(3B_i - B_i\right) = B_i$$

$$\rightarrow \nabla\times\left[\tfrac{1}{2}\vec{B}\times\vec{r}\right] = \vec{B}$$ (for uniform $\vec{B}$ indep. of $\vec{r}$)

The Hamiltonian for an otherwise free particle in a magnetic field is given by substitution of canonical momentum $\vec{p}$ with the kinetic momentum $(\vec{p} - q\vec{A})$.

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} - q\hat{\vec{A}}\right)^2 = \frac{\hat{\vec{p}}^2}{2m} + \frac{q^2}{2m}\hat{\vec{A}}^2 - \frac{q}{m}\,\frac{\hat{\vec{p}}\cdot\hat{\vec{A}} + \hat{\vec{A}}\cdot\hat{\vec{p}}}{2}$$

$$\hat H = \frac{\hat{\vec p}^{\,2}}{2m} + \frac{q^2}{2m}\left[\frac12 \vec B\times\hat{\vec r}\right]^2 - \underbrace{\frac{q}{2m}\left[\hat{\vec p}\cdot\frac12\vec B\times\hat{\vec r} + \frac12\vec B\times\hat{\vec r}\cdot\hat{\vec p}\right]}_{\hat H_{int}}$$

$$\hat H_{int} = -\frac{q}{4m}\epsilon_{ijk}\left[\hat p_i B_j \hat x_k + B_i \hat x_j \hat p_k\right]$$

dummy indices $i \leftrightarrow j$

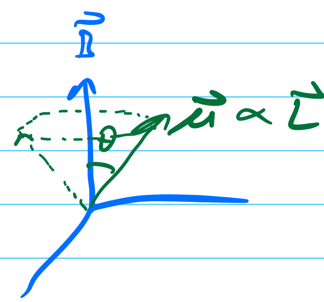
$$= -\frac{q}{4m}\epsilon_{ijk} B_i\left[\underbrace{-\hat p_j \hat x_k + \hat x_j \hat p_k}_{\epsilon_{jkl}L_l}\right]$$

$$= -\frac{q}{4m}\underbrace{\epsilon_{ijk} B_i \epsilon_{jkl}}_{\epsilon_{ijk}\epsilon_{jkl} = 2\delta_{il}} L_l$$

$$= -\frac{q}{2m}\vec B\cdot\hat{\vec L} \equiv -\hat{\vec\mu}\cdot\vec B$$

Magnetic moment

$$\boxed{\hat{\vec\mu} = \frac{q}{2m}\hat{\vec L}}$$



Torque $\left|\frac{d\vec L}{dt}\right| = \left|\vec\mu\times\vec B\right| = \frac{|q|}{2m}LB\sin\theta$

$$\left|\frac{1}{L\sin\theta}\frac{d\vec L}{dt}\right| = \frac{d\phi}{dt} \equiv \omega$$

Radius of orbit of precession of $\vec\mu$

Angular precession speed $\omega = \frac{|q|}{2m}B$

$$\mu_z = \frac{q}{2m}L_z \to \frac{q}{2m}\hbar\, m_\ell$$

$\hat L_z$ quantum number

For an electron,

$$\mu_z = \frac{e\hbar}{2m} m_\ell \equiv \mu_B m_\ell \quad , \quad m_\ell = -\ell, \dots, +\ell$$

$$\mu_B = \text{Bohr Magneton} = \frac{e\hbar}{2m}$$

Spin: $s = 1/2$, $m_s = \pm 1/2$

$$\hat{\vec{S}}^2 \to \hbar^2 s(s+1) = \frac{3}{4}\hbar^2$$

$$\hat{S}_z \to \hbar m_s = \pm \frac{1}{2}\hbar$$

Particle with both $\vec{L}$ and $\vec{S}$:

$$\vec{S}_{tot} = \sum_i \vec{S}_i \quad , \quad \vec{L}_{tot} = \sum_i \vec{L}_i$$

$$\vec{J}_{tot} = \vec{L}_{tot} + \vec{S}_{tot}.$$

$$[\hat{\vec{L}}, \hat{\vec{S}}] = 0$$ → simultaneously diagonalize $\hat{\vec{L}}^2, \hat{L}_z, \hat{\vec{S}}^2, \hat{S}_z$

States $|\alpha, \ell, m\rangle \otimes |s, m_s\rangle$

labels quantum numbers of other mutually commuting observables

$$\langle \vec{r}\, m_s | \psi \rangle = \psi_{m_s}(\vec{r}) \doteq \begin{pmatrix} \psi_\uparrow(\vec{r}) \\ \psi_\downarrow(\vec{r}) \end{pmatrix} \begin{array}{l} \leftarrow m_s = +1/2 \\ \leftarrow m_s = -1/2 \end{array}$$

Factorized states: $\psi = \psi_{\ell m}(\vec{r}) \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$

Pauli spinor

Normalize the spinor $|c_\uparrow|^2 + |c_\downarrow|^2 = 1$

Normalize the wavefunction $\int d^3x \, |\psi_{\ell m}(\vec{r})|^2 = 1$

Eigenstates of $\hat{L}^2$ and $\hat{L}_z$ have wavefunctions

$$\psi_{\ell m}(r, \theta, \phi) = \psi_\ell(r) \, Y_{\ell m}(\theta, \phi)$$

so, the spin-1/2 particle in state $|\alpha, \ell m\rangle \otimes |\uparrow\rangle$

has wavefunction

$$\psi = \psi_\ell(r) \, Y_{\ell m}(\theta, \phi) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $|\alpha, \ell m\rangle \otimes |\downarrow\rangle$ has wavefunction

$$\psi = \psi_\ell(r) \, Y_{\ell m}(\theta, \phi) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Properties of spherical harmonics:

- $Y_{\ell m}(\theta,\phi) = P_{\ell,m}(\cos\theta) \cdot \frac{1}{\sqrt{2\pi}} e^{im\phi}$

  ↑ Associated Legendre polynomials

- $P_{\ell,m}(w) = (-1)^m \sqrt{\frac{(\ell-m)!}{(\ell+m)!} \frac{(2\ell+1)}{2}} \frac{1}{\ell!} \frac{1}{2^\ell}$

  $\times (1-w^2)^{m/2} \left(\frac{d}{dw}\right)^{\ell+m} (w^2-1)^\ell$

- $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$

- $m=0$: $Y_{\ell,0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$

- Orthogonality: $\int_{-1}^{1} dw\, P_\ell(w) P_{\ell'}(w) = \frac{2}{2\ell+1} \delta_{\ell\ell'}$

- Completeness: $\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\vec{n}') Y_{\ell m}(\vec{n}) = \delta(\vec{n}'-\vec{n})$

  $= \delta(\cos\theta - \cos\theta') \delta(\phi-\phi')$

- Orthogonality: $\int d\Omega_{\vec{n}}\, Y^*_{\ell' m'}(\vec{n}) Y_{\ell m}(\vec{n}') = \frac{2\ell+1}{4\pi} P_\ell(\vec{n}\cdot\vec{n}')$

Rotation of Spinors with $s = 1/2$

Given eigenstates of $\hat{S}_z$ with spinors $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$,

we would like to find eigenstates of $\vec{S}\cdot\vec{n}$ for arbitrary unit vector $\vec{n}$.

The unitary operator that implements a rotation by angle ϕ about axis $\vec{n}$, acting on a spinor state, is:

$$g(\vec{n},\phi) = \exp\left(-i\frac{\vec{S}\cdot\vec{n}}{\hbar}\phi\right)$$

$$= \exp\left(-i\frac{\vec{\sigma}\cdot\vec{n}}{2}\phi\right)$$

Using $\sigma_i^2 = \mathbb{1}$, $\vec{n}^2 = 1$, $(\vec{\sigma}\cdot\vec{n})^2 = \mathbb{1}$,

$$(\vec{\sigma}\cdot\vec{n})^m = \begin{cases} \mathbb{1}, & m \text{ even} \\ \vec{\sigma}\cdot\vec{n}, & m \text{ odd} \end{cases}$$

Then, $g(\vec{n},\phi) = \cos\frac{\phi}{2}\,\mathbb{1} - i\sin\frac{\phi}{2}\,(\vec{\sigma}\cdot\vec{n})$

The spin operator rotates as a vector:

$$\sigma_i' \equiv g^{-1}\sigma_i g = \sum_j R_{ij}\,\sigma_j$$

rotation matrix

Example: Rotate σ_1 about $\vec{n} = \vec{e}_3$:

$$\sigma_1' = e^{i\sigma_3\phi/2}\,\sigma_1\,e^{-i\sigma_3\phi/2}$$

$$= \left(\cos\tfrac{\phi}{2}\,\mathbb{1} + i\sin\tfrac{\phi}{2}\,\sigma_3\right)\sigma_1\left(\cos\tfrac{\phi}{2}\,\mathbb{1} - i\sin\tfrac{\phi}{2}\,\sigma_3\right)$$

$$= \sigma_1\cos^2\phi/2 + i\sin\phi/2\,\cos\phi/2\,[\sigma_3,\sigma_1] + \sin^2\phi/2\,\sigma_3\sigma_1\sigma_3$$

$$= \sigma_1\cos^2\phi/2 + \tfrac{i}{2}\sin\phi\,(i\sigma_2) + \sin^2\phi/2\,\Big(\underbrace{\sigma_3^2\sigma_1}_{\mathbb{1}\sigma_1} + \underbrace{\sigma_3[\sigma_1,\sigma_3]}_{\sigma_3(-2i\sigma_2)\,=\,-2\sigma_1}\Big)$$

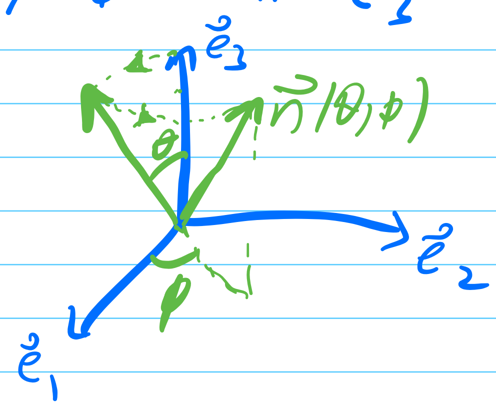
$$= \sigma_1\left(\cos^2\phi/2 - \sin^2\phi/2\right) - \tfrac{1}{2}\sin\phi\,(2\sigma_2)$$

$$= \sigma_1\cos\phi - \sigma_2\sin\phi$$

Compare with $\vec{V} = V\vec{e}_1 \xrightarrow{R} V\cos\phi\,\vec{e}_1 - V\sin\phi\,\vec{e}_2$

To construct the eigenspinor of $\frac{\vec{\sigma}}{2}\cdot\hat{n}(\theta,\phi)$ with eigenvalue $+1/2$, for example:

First rotate eigenstate of σ_3 by θ about $\vec{e}_2$, then by ϕ about $\vec{e}_3$



$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \underset{\substack{\uparrow\\ g(\hat{e}_1,\,\phi)}}{e^{-i\sigma_3\phi/2}}\ \underset{\substack{\uparrow\\ g(\hat{e}_2,\,\theta)}}{e^{-i\sigma_2\theta/2}} \begin{pmatrix}1\\0\end{pmatrix}$$

$$= \left(\cos\phi/2 - i\sin\phi/2\,\sigma_3\right)\left(\cos\theta/2 - i\sin\theta/2\,\sigma_2\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \left[\cos\theta/2 \begin{pmatrix} e^{-i\phi/2} & 0\\ 0 & e^{i\phi/2}\end{pmatrix} + \sin\theta/2 \begin{pmatrix}0 & e^{-i\phi/2}\\ e^{i\phi/2} & 0\end{pmatrix}\right]\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \boxed{\begin{pmatrix}\cos\theta/2\; e^{-i\phi/2}\\ \sin\theta/2\; e^{i\phi/2}\end{pmatrix}} \equiv |\uparrow\rangle'$$

- satisfies $\frac{\vec{\sigma}}{2}\cdot\vec{n}\,|\uparrow\rangle' = \frac{1}{2}|\uparrow\rangle'$

Check! $\theta = \pi/2$, $\phi = 0 \rightarrow |\uparrow\rangle' = |\uparrow\rangle_x$

spin-up about $\hat{e}_x$

$$|\uparrow\rangle_x = \begin{pmatrix}\cos\pi/4\\ \sin\pi/4\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\frac{\sigma_x}{2}|\uparrow\rangle_x = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}|\uparrow\rangle_x \quad ✓$$