# PHYS 621 Lecture Notes 18

N14.6 **Matrix Representation of Angular Momentum Operators**

We have calculated

$$\| J_+ |j,m\rangle \|^2 = \hbar^2 [ j(j+1) - m^2 - m ]$$

Since we also have $J_+ |j,m\rangle \propto |j,m+1\rangle$

for $m+1 \leq j$,

$$J_+ |j,m\rangle = \alpha_{j,m+1} \; |j,m+1\rangle$$

where $\alpha_{j,m+1} = \sqrt{j(j+1) - m^2 - m}$

$$= \sqrt{(j-m)(j+m+1)} \; ,$$

or, $\boxed{\alpha_{j,m} = \sqrt{(j+m)(j-m+1)}}$

Matrix elements of $J_+$:

$$\boxed{(J_+)_{jm',jm}} \equiv \langle jm' | J_+ | j\, m \rangle$$

$$\boxed{= \hbar\, \alpha_{j,m+1} \; \delta_{m',m+1}} \qquad m+1 \leq j$$

Similarly, using $J_- |jm\rangle \propto |j,m-1\rangle$

and $\| J_- |jm\rangle \|^2 = \hbar^2 [ j(j+1) - m^2 + m ]$,

$$\boxed{(J_-)_{jm',jm} = \hbar\, \alpha_{jm} \; \delta_{m',m-1}} \qquad m-1 \geq -j$$

From the definitions $J_\pm = J_1 \pm i J_2$,

$$J_1 = \frac{1}{2}(J_+ + J_-) \quad \text{and} \quad J_2 = \frac{1}{2i}(J_+ - J_-)$$

Hence,

$$\boxed{\begin{aligned} (J_1)_{jm',jm} &= \frac{\hbar}{2}\left(\alpha_{j,m+1}\, \delta_{m',m+1} + \alpha_{jm}\, \delta_{m',m-1}\right) \\ (J_2)_{jm',jm} &= \frac{\hbar}{2i}\left(\alpha_{j,m+1}\, \delta_{m',m+1} - \alpha_{jm}\, \delta_{m',m-1}\right) \end{aligned}}$$

Also,

$$(J_3)_{jm',jm} = \hbar m\, \delta_{m',m}$$

$$(\vec{J}^2)_{jm',jm} = j(j+1)\hbar^2 \delta_{m',m}$$

Example: $j = 1/2$, $m = +1/2, -1/2$

$$\alpha_{1/2,1/2} = \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)} = 1$$

$$\alpha_{1/2,-1/2} = \sqrt{\left(\frac{1}{2}-\frac{1}{2}\right)\left(\frac{1}{2}+\frac{1}{2}+1\right)} = 0$$

$$\Rightarrow \quad J_1^{(1/2)} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_1$$

$$J_2^{(1/2)} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_2$$

$$J_3^{(1/2)} = \hbar\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_3$$

$\sigma_1, \sigma_2, \sigma_3 \equiv$ Pauli $\sigma$-matrices.

Exercise: For $j=1$, $m=+1, 0, -1$

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

## Composition of Angular Momentum

Consider a system of several objects, each with angular momentum.

There are eigenstates of $\vec{J}^2$ and $J_z$ for each object separately.

Alternatively, we can consider eigenstates of the total $\vec{J}^2$ and total $J_z$.

We would like to understand the relationship between these different descriptions of the system.

Consider a system of two angular momenta $\vec{J}_1, \vec{J}_2$, and total $\vec{J} = \vec{J}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \vec{J}_2 \equiv \vec{J}_1 + \vec{J}_2$

Eigenstates of $\vec{J}_1^2, \vec{J}_2^2, J_{1z}, J_{2z}$:

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle \equiv |j_1, j_2\ m_1, m_2\rangle$$

Algebra: $[J_{1i}, J_{1j}] = i\hbar \sum_k \epsilon_{ijk} J_{1k}$

$[J_{2i}, J_{2j}] = i\hbar \sum_k \epsilon_{ijk} J_{2k}$

$[J_{1i}, J_{2j}] = 0$

$[J_1^2, J_{1z}] = 0 = [J_2^2, J_{2z}]$

$\vec{J}_1^2 |j_1 j_2 \; m_1 m_2\rangle = \hbar^2 j_1(j_1+1) |j_1 j_2 \; m_1 m_2\rangle$

$\vec{J}_2^2 |j_1 j_2 \; m_1 m_2\rangle = \hbar^2 j_2(j_2+1) |j_1 j_2 \; m_1 m_2\rangle$

$J_{1z} |j_1 j_2 \; m_1 m_2\rangle = \hbar m_1 |j_1 j_2 \; m_1 m_2\rangle$

$J_{2z} |j_1 j_2 \; m_1 m_2\rangle = \hbar m_2 |j_1 j_2 \; m_1 m_2\rangle$

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Alternative basis: Keep $j_1, j_2$

Replace $m_1, m_2$ with total $j, m$.

Check that $\vec{J}^2, J_z, \vec{J}_1^2, \vec{J}_2^2$ are mutually commuting:

$[\vec{J}^2, \vec{J}_1^2] = [\vec{J}_1^2 + \vec{J}_2^2 + 2\vec{J}_1 \cdot \vec{J}_2, \vec{J}_1^2]$

$= [\vec{J}_1^2 + \vec{J}_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}, \vec{J}_1^2]$

$= 0$ (because $[J_{1z}, \vec{J}_1^2] = [J_{1+}, \vec{J}_1^2] = [J_{2+}, \vec{J}_1^2]$
$= 0$)

Similarly,

$[\vec{J}^2, \vec{J}_2^2] = 0$

$$[J_z, \vec{J}_1^{\,2}] = [J_{1z} + J_{2z}, \vec{J}_1^{\,2}] = 0$$

(because $[J_{1z}, \vec{J}_1^{\,2}] = 0$)

$$[J_z, \vec{J}_2^{\,2}] = 0$$

However, $[\vec{J}^2, J_{1z}] \neq 0$ (because $[J_{1z}, J_{1\pm}] \neq 0$)

and $[\vec{J}^2, J_{2z}] \neq 0$

$\Rightarrow$ Can specify $j_1 j_2 m_1 m_2$ or $j_1 j_2 j m$

but not $m_1 m_2 j m$ simultaneously

Total angular momentum basis:

$$\vec{J}_1^{\,2} |j_1 j_2 \; j m\rangle = \hbar^2 j_1(j_1+1) |j_1 j_2 \; j m\rangle$$

$$\vec{J}_2^{\,2} |j_1 j_2 \; j m\rangle = \hbar^2 j_2(j_2+1) |j_1 j_2 \; j m\rangle$$

$$\vec{J}^2 |j_1 j_2 \; j m\rangle = \hbar^2 j(j+1) |j_1 j_2 \; j m\rangle$$

$$J_z |j_1 j_2 \; j m\rangle = \hbar m |j_1 j_2 \; j m\rangle$$

Using completeness $\mathbb{1} = \sum_{m_1, m_2} |j_1 j_2 \; m_1 m_2\rangle\langle j_1 j_2 \; m_1 m_2|$

we can write $|j_1 j_2 \; j m\rangle$ as:

$$|j_1 j_2 \; j m\rangle = \sum_{m_1, m_2} |j_1 j_2 \; m_1 m_2\rangle \underbrace{\langle j_1 j_2 m_1 m_2 | j_1 j_2 \; j m\rangle}_{\text{Clebsch-Gordan coefficients}}$$

## Properties of Clebsch-Gordan Coefficients:

(1) Since $J_z = J_{1z} + J_{2z}$,

$$\langle j_1 j_2 m_1 m_2 | (J_z - J_{1z} - J_{2z}) | j_1 j_2 j m \rangle$$

$$= \hbar (m - m_1 - m_2) \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle$$

$\Rightarrow$ Clebsch-Gordan coefficients vanish unless $\boxed{m = m_1 + m_2}$

(2) This is more subtle, but Clebsch-Gordan coefficients vanish unless

$$\boxed{|j_1 - j_2| \leq j \leq j_1 + j_2}$$

For evidence, the number of states (i.e. values of $m$) with fixed $j$ is $2j+1$. $(m = -j, -j+1, \ldots, j-1, j)$

Suppose $j_1 \geq j_2$.

$j_{max} \rightarrow$ $j_1+j_2$

$j_{min} \rightarrow$ $j=j_1-j_2$

$$\sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = \frac{2}{2}\left[(j_1+j_2+1)(j_1+j_2) - (j_1-j_2-1)(j_1-j_2)\right] + \left[(j_1+j_2) - (j_1-j_2) + 1\right]$$

$$= (2j_1+1)(2j_2+1)$$

= total # states in $m_1, m_2$ basis

Example: $j_1 = 1/2,\ j_2 = 1/2$

$$\tfrac{1}{2} - \tfrac{1}{2} \le j \le \tfrac{1}{2} + \tfrac{1}{2} \rightarrow j = 0, 1$$

Nonvanishing Clebsch-Gordan coefficients: $m = m_1 + m_2$
Label states by $m_1 = \pm 1/2,\ m_2 = \pm 1/2$ or $j, m$

$$\langle m_1 m_2 | j\, m \rangle$$

$$J_+ |j=0, m=0\rangle = (J_{1+} + J_{2+}) \Big[ |\overset{m_1}{1/2}, \overset{m_2}{-1/2}\rangle \langle \overset{m_1}{\tfrac{1}{2}}, \overset{m_2}{-1/2} | \overset{j}{0}\ \overset{m}{0} \rangle + |-1/2, 1/2\rangle \langle -1/2, 1/2 | 0\ 0\rangle \Big]$$

$$0 = \sqrt{\left(\tfrac{1}{2} - (-1/2)\right)\left(\tfrac{1}{2} + (-1/2) + 1\right)}\ \hbar\ |1/2, 1/2\rangle \langle 1/2, -1/2 | 0\ 0\rangle$$
$$+ \sqrt{\tfrac{1}{2} - (-1/2)\ (1/2 + (-1/2) + 1)}\ \hbar\ |1/2, 1/2\rangle \langle -1/2, 1/2 | 0\ 0\rangle$$
$$= \hbar\, |1/2, 1/2\rangle \Big( \langle 1/2, -1/2 | 0\ 0\rangle + \langle -1/2, 1/2 | 0\ 0\rangle \Big)$$

$$\rightarrow \langle 1/2, -1/2 | 0\ 0\rangle = -\langle -1/2, 1/2 | 0\ 0\rangle$$

$\Rightarrow$ with $j_1 = 1/2,\ j_2 = 1/2$:

$$\boxed{|j=0, m=0\rangle = \underset{\text{Normalization}}{\frac{1}{\sqrt{2}}} \Big( |m_1 = 1/2, m_2 = -1/2\rangle - |m_1 = -1/2, m_2 = +1/2\rangle \Big)}$$

$$\langle \tfrac{1}{2} -1/2 | 0\ 0\rangle = -\langle -1/2\ 1/2 | 0\ 0\rangle = \frac{1}{\sqrt{2}}$$

If these are states of spin-½ particles, we have found a state with total $j=0$, $m=0$:

$$\boxed{|j=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}$$

This state often appears in discussions of entanglement, as it is a prototypical entangled state of two particles.

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By acting with $J_+$ and $J_-$ on $|j=1, m=\pm 1, 0\rangle$ states, we could determine the $j=1$ Clebsch-Gordan coefficients.

Result for $j_1 = ½$, $j_2 = ½$:

$$|j=1, m=1\rangle = |\uparrow\uparrow\rangle$$

$$|j=1, m=0\rangle = \underbrace{\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)}_{m=0 \text{ state orthogonal to } |j=0, m=0\rangle}$$

$$|j=1, m=-1\rangle = |\downarrow\downarrow\rangle$$