PHYS621 Lecture Notes 17

N14.5

Angular Momentum

To analyze quantum-mechanical systems in 3D, it will be useful to consider angular momentum in quantum mechanics.

Classically: $\vec{L} = \vec{x} \times \vec{p}$, or $L_i = \sum_{j,k=1}^{3} \epsilon_{ijk} x_j p_k$

ϵ_{ijk} is the Levi-Civita symbol

$\epsilon_{123} = +1$

ϵ_{ijk} is completely antisymmetric on exchange of its indices.

For example, $L_z \equiv L_3 = \epsilon_{312} x_1 p_2 + \epsilon_{321} x_2 p_1$

$$= x_1 p_2 - x_2 p_1$$

$$\equiv x p_y - y p_x$$

Quantum mechanics: $\hat{L}_i = \sum_{j,k=1}^{3} \epsilon_{ijk} \hat{x}_j \hat{p}_k$

Consider the commutator $[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z]$

$$[\hat{L}_x, \hat{L}_y] = \hat{y}[\hat{p}_z, \hat{z}]\hat{p}_x + [\hat{z}, \hat{p}_z]\hat{p}_y \hat{x}$$

$$= -i\hbar \hat{y}\hat{p}_x + i\hbar \hat{p}_y \hat{x} \quad = i\hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)$$

$$\boxed{[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z}$$

More generally, $\boxed{[\hat{L}_m, \hat{L}_n] = i\hbar \sum_{r=1}^{3} \epsilon_{mnr} \hat{L}_r}$

Consider the operator $\hat{\vec{L}}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 = \sum_n \hat{L}_n^2$

$$[\hat{L}_m, \hat{\vec{L}}^2] = \sum_n [\hat{L}_m, \hat{L}_n]\hat{L}_n + \hat{L}_n[\hat{L}_m, \hat{L}_n]$$

$$= \sum_n \epsilon_{mnr} \hat{L}_r \hat{L}_n + \sum_n \epsilon_{mnr} \hat{L}_n \hat{L}_r$$

$$\boxed{[\hat{L}_m, \hat{\vec{L}}^2] = 0}$$

(because $\epsilon_{mnr} = -\epsilon_{mrn}$ but $(\hat{L}_r\hat{L}_n + \hat{L}_n\hat{L}_r)$ is symmetric in $n \leftrightarrow r$)

N 14.6 Hence, both $\hat{L}^2$ and one component of $\hat{\vec{L}}$ can be known simultaneously

→ Can include in description of states the eigenvalues of the operators $\hat{L}^2$ and $\hat{L}_z$, for example.

We need the eigenvalues of the operators $\hat{L}_z$ and $\hat{L}^2$.

Define $\hat{L}_\pm \equiv \hat{L}_1 \pm i\hat{L}_2$, $\hat{L}_z \equiv \hat{L}_3$

$$[\hat{L}_z, \hat{L}_+] = [\hat{L}_z, \hat{L}_1 + i\hat{L}_2]$$

$$= i\hbar(\hat{L}_2 - i\hat{L}_1) = \hbar(\hat{L}_1 + i\hat{L}_2)$$

$$= \hbar \hat{L}_+$$

$$[\hat{L}_z, \hat{L}_-] = [\hat{L}_z, \hat{L}_1 - i\hat{L}_2]$$

$$= i\hbar(\hat{L}_2 + i\hat{L}_1) = -\hbar(\hat{L}_1 - i\hat{L}_2)$$

$$= -\hbar \hat{L}_-$$

$$[\hat{J}_+, \hat{J}_-] = [\hat{L}_1 + i\hat{L}_2, \hat{L}_1 - i\hat{L}_2] = -i[\hat{L}_1, \hat{L}_2] + i[\hat{L}_2, \hat{L}_1]$$

$$= -i(2i\hbar \hat{L}_z) = 2\hbar \hat{L}_z$$

Label states by $\hat{L}^2$, $\hat{L}_z$ eigenvalues, possibly other quantum numbers in addition:

$$\hat{L}^2 |\lambda^2, m, \ldots\rangle = \underbrace{\hbar^2 \lambda^2}_{\hat{L}^2 \text{ eigenvalue } \hbar^2\lambda^2} |\lambda^2, m, \ldots\rangle$$

$$\hat{L}_z |\lambda^2, m, \ldots\rangle = \underbrace{\hbar m}_{\hat{L}_z \text{ eigenvalue } \hbar m.} |\lambda^2, m, \ldots\rangle$$

Since $\hat{L}_i$ is Hermitian,

$$\langle \lambda^2, m, \ldots | \hat{L}^2 | \lambda^2, m, \ldots \rangle = \sum_n \| \hat{L}_n | \lambda^2, m, \ldots \rangle \|^2$$

$$= \lambda^2 \hbar^2 \geq 0$$

$$\rightarrow \boxed{\lambda^2 \geq 0}$$

Consider $\langle \lambda^2, m, \ldots | (\hat{L}_x^2 + \hat{L}_y^2) | \lambda^2, m, \ldots \rangle \quad \geq 0$

$$= \langle \lambda^2, m, \ldots | (\hat{L}^2 - \hat{L}_z^2) | \lambda^2, m, \ldots \rangle$$

$$= \hbar^2 (\lambda^2 - m^2)$$

$$\Rightarrow \boxed{\lambda^2 - m^2 \geq 0}$$

The operator $\hat{L}_+$ acts as a raising operator of $\hat{L}_z$:

$$\hat{L}_z (\hat{L}_+ |\lambda^2, m, \ldots\rangle) = (\hat{L}_+ \hat{L}_z + [\hat{L}_z, \hat{L}_+]) |\lambda^2, m, \ldots\rangle$$

$$= (m+1)\hbar (\hat{L}_+ |\lambda^2, m, \ldots\rangle)$$

Similarly, $\hat{L}_z (\hat{L}_- |\lambda^2, m, \ldots\rangle) = (m-1)\hbar (\hat{L}_- |\lambda^2, m, \ldots\rangle)$

Since $m^2 \le \lambda^2$, $\hat{L}_+$ can't increase the $\hat{L}_z$ eigenvalue above $\hbar\lambda$
and $\hat{L}_-$ can't decrease the $\hat{L}_z$ eigenvalue below $-\hbar\lambda$.

$\Rightarrow \exists\, m_{max}$ such that $\hat{L}_+ |\lambda^2, m_{max}, \ldots\rangle = 0$

and $\exists\, m_{min}$ such that $\hat{L}_- |\lambda^2, m_{min}, \ldots\rangle = 0$

We can find additional restrictions on the eigenvalues:

Consider $\hat{L}_- \hat{L}_+ = \hat{L}_+^\dagger \hat{L}_+ = \hat{L}_1^2 + \hat{L}_2^2 + i[\hat{L}_1, \hat{L}_2]$
$= \hat{\vec{L}}^2 - \hat{L}_z^2 - \hbar \hat{L}_z$

$$\langle \lambda^2, m, \ldots | \hat{L}_+^\dagger \hat{L}_+ | \lambda^2, m, \ldots\rangle \ge 0$$

$$= \boxed{\hbar^2(\lambda^2 - m^2 - m) \ge 0}$$

Similarly, $\hat{L}_+ \hat{L}_- = \hat{L}_-^\dagger \hat{L}_- = \hat{\vec{L}}^2 - \hat{L}_z^2 + \hbar \hat{L}_z$

$$\langle \lambda^2, m, \ldots | \hat{L}_-^\dagger \hat{L}_- | \lambda^2, m, \ldots\rangle \ge 0$$

$$= \boxed{\hbar^2(\lambda^2 - m^2 + m) \ge 0}$$

For $m = m_{max}$, $\hat{L}_+ |\lambda^2, m_{max}, \ldots\rangle = 0 \Rightarrow \boxed{\lambda^2 - m_{max}^2 - m_{max} = 0}$

For $m = m_{min}$, $\hat{L}_- |\lambda^2, m_{min}, \ldots\rangle = 0 \Rightarrow \boxed{\lambda^2 - m_{min}^2 + m_{min} = 0}$

$$m_{max}(m_{max}+1) = m_{min}(m_{min}-1)$$

$$\Rightarrow \boxed{m_{max} = -m_{min}}$$

Hence, $-m_{max} \le m \le m_{max}$

Since m changes by ± 1 through $\hat{L}_\pm$, there must be an integer number of steps between $-m_{max}$ and m_{max}:

$$2m_{max} = n \in \mathbb{N}$$

non-negative integers (natural numbers)

$$m_{max} = \frac{n}{2} \equiv \ell \quad \text{— half-integer}$$

Since $\lambda^2 - m_{max}(m_{max}+1) = 0$

$$\rightarrow \boxed{\lambda^2 = \ell(\ell+1)}$$

To summarize, we can replace the quantum number λ^2 with ℓ:

$$\begin{cases} \hat{L}_z |\ell, m, \dots\rangle = m\hbar |\ell, m, \dots\rangle \\ \hat{L}^2 |\ell, m, \dots\rangle = \ell(\ell+1)\hbar^2 |\ell, m, \dots\rangle \\ -\ell \leq m \leq \ell \text{, divided in integer steps:} \\ m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\} \end{cases}$$

So far we have considered the orbital angular momentum of a particle, but we can consider more complicated systems with different contributions to the angular momentum including <u>spin</u>.

The total angular momentum satisfies the same algebra:

$$[\hat{J}_r, \hat{J}_s] = i\hbar\, \epsilon_{rst} \hat{J}_t$$

Hence, the same analysis as for $\hat{L}$ applies, and the total angular momentum satisfies

$$\hat{J}^2 |j, m, \dots\rangle = \hbar^2 j(j+1) |j, m, \dots\rangle$$

$$\hat{J}_z |j, m, \dots\rangle = \hbar m |j, m, \dots\rangle, \quad m \in \{-j, -j+1, \dots, j\}$$