PHYS 621 Lecture Notes 15

Feynman Path Integral and Propagators

N 10.1 Consider the **propagator** = time-evolution operator in the coordinate basis.

$$U(x',t';x,t) \equiv {}_s\langle x'|\hat{U}(t',t)|x\rangle_s$$

$$= {}_s\langle x'|\hat{U}(t',t_0)\,\hat{U}(t_0,t)|x\rangle_s$$

$$= \underbrace{{}_s\langle x'|\hat{U}(t',t_0)}_{{}_H\langle x',t'|}\ \underbrace{\hat{U}(t,t_0)^{-1}|x_s\rangle}_{|x,t\rangle_H}$$

$$= {}_H\langle x',t'|x,t\rangle_H$$

The time-evolution operator is

$$\hat{U}(t,t_0) = \sum_{n,a} |E_n,a\rangle_s\langle E_n,a|\, e^{-iE_n(t-t_0)/\hbar}$$

$$U(x',t';x,t) = \sum_{n,a} {}_s\langle x'|E_n,a\rangle\langle E_n,a|x\rangle_s\, e^{-iE_n(t'-t)/\hbar}$$

$$\boxed{\lim_{t'\to t} U(x',t';x,t)} = \sum_{n,a} {}_s\langle x'|E_n,a\rangle\langle E_n,a|x\rangle_s$$

$$= {}_s\langle x'|x\rangle_s$$

$$\boxed{= \delta(x-x')}$$

For $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, $U(x,t;x_0 t_0)$ satisfies for $t > t_0$

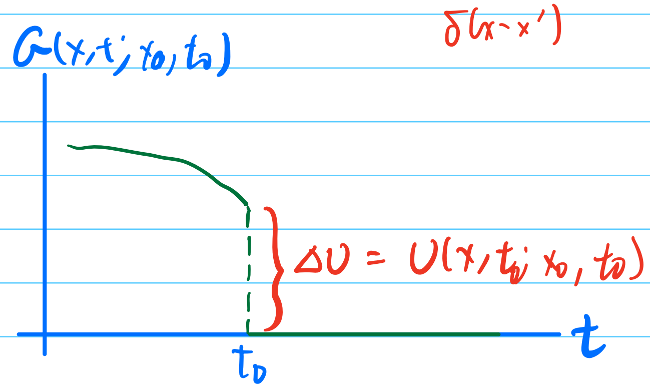
$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) - i\hbar\frac{\partial}{\partial t}\right)U(x,t;x_0,t_0) = 0$$

If we define $G(x,t;x_0,t_0) = \begin{cases} 0 & \text{for } t < t_0 \\ U(x,t;x_0,t_0) & \text{for } t \geq t_0 \end{cases}$

then

$$i\hbar\frac{\partial}{\partial t}G(x,t;x_0,t_0) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)(x,t;x_0,t_0)$$

$$+ i\hbar\,\underbrace{G(x,t_0;x_0,t_0)}_{\delta(x-x')}\,\delta(t-t_0)$$



$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) - i\hbar\frac{\partial}{\partial t}\right)G(x,t;x_0,t_0) = -i\hbar\,\delta(x-x_0)\,\delta(t-t_0)$$

$\to$ $G(x,t;x_0,t_0)$ is a Green's function for the Schrödinger operator.

The Heisenberg-picture position eigenstates:

$$\hat{x}_H(t)\,|x,t\rangle_H = x(t)\,|x,t\rangle_H$$

$$\hat{x}_H(t) = e^{i\hat{H}t/\hbar}\,\hat{x}_S\,e^{-i\hat{H}t/\hbar}$$

$$|x\rangle_S = e^{-i\hat{H}t/\hbar}\,|x,t\rangle_H$$

We are now ready to develop a path-integral representation for the propagator.

Divide the interval $[t, t')$ into $n+1$ intervals of length

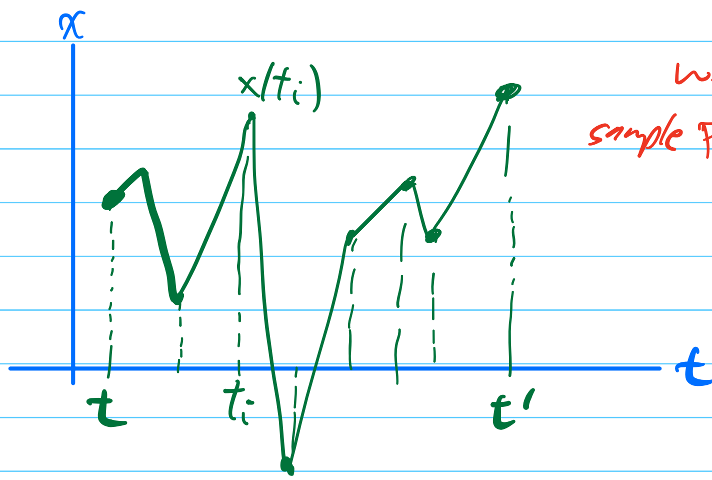
$$\epsilon = \frac{t'-t}{n+1}$$

$$t_0 = t, \quad t_1 = t+\epsilon, \ldots, \quad t_{n+1} = t'$$

Completeness: At any time t, $\int dx_i \, |x_i, t_i\rangle\langle x_i, t_i| = \mathbb{1}$

warning: something this is called the propagator; sometimes $\frac{1}{i\hbar}G(x',t';x,t)$ is called the propagator.

$$U(x',t';x,t) = \int dx_1 \cdots dx_n \, \langle x',t' | x_n, t_n\rangle \langle x_n, t_n | x_{n-1}, t_{n-1}\rangle \cdots \langle x_1, t_1 | x, t\rangle$$



we integrate over $x(t_i)$, so sample paths are not differentiable.

We calculate $\,_H\langle x(t_i), t_i | x(t_{i-1}), t_{i-1}\rangle_H$

$$= \,_S\langle x(t_i) | e^{-i\epsilon \hat{H}/\hbar} | x(t_{i-1})\rangle_S$$

$$= \int dp(t_i) \, \underbrace{\langle x(t_i) | p(t_i)\rangle}_{\frac{e^{i p_i x_i/\hbar}}{\sqrt{2\pi\hbar}}} \underbrace{\langle p(t_i) | e^{-i\epsilon \hat{H}/\hbar} | x(t_{i-1})\rangle}$$

Schrödinger picture state

$$= \langle p(t_i) | (\mathbb{1} - i\epsilon\hat{H}/\hbar) | x(t_{i-1})\rangle$$

$$= \langle p(t_i) | x(t_{i-1})\rangle (1 - i\epsilon H(p_i, x_{i-1})/\hbar)$$

$$\simeq \frac{e^{-i p_i x_{i-1}/\hbar}}{\sqrt{2\pi\hbar}} e^{-i\epsilon H(p_i, x_{i-1})/\hbar}$$

$$= \int \frac{dp(t_i)}{2\pi\hbar} e^{i p_i (x_i - x_{i-1})/\hbar} e^{-i\epsilon H(p_i, x_{i-1})/\hbar}$$

$$\boxed{U(x',t';x,t)} = \int dx(t_1)\cdots dx(t_n)\int dp(t_1)\cdots dp(t_n)$$

$$\exp\Big\{\frac{i}{\hbar}\Big[p(t')\big(x(t')-x(t_n)\big)+\cdots$$

$$+\,p(t_1)\big(x(t_1)-x(t)\big)$$

$$-\epsilon\Big(H\big(p(t'),x(t_n)\big)+\cdots+H\big(p(t_1),x(t)\big)\Big)\Big]\Big\}$$

$$\boxed{\equiv \int \mathcal{D}x(t)\int\mathcal{D}p(t)\,\exp\Big\{\frac{i}{\hbar}\int_t^{t'}d\tau\Big[p(\tau)\,\dot{x}(\tau)-H\big(p(\tau),x(\tau)\big)\Big]\Big\}}$$

$$\dot{x}\,dt = x(t_{i+1})-x(t_i)$$

- Path integral in phase space

We can evaluate the integrals over $p(t_i)$ explicitly:

$$\int dp_i\,\exp\Big[\frac{i}{\hbar}\Big(p_i(x_i-x_{i-1})-\epsilon H(p_i,x_{i-1})\Big)\Big]$$

$$=\int dp_i\,\exp\Big[\frac{i}{\hbar}\Big(p_i(x_i-x_{i-1})-\epsilon\frac{p_i^2}{2m}-\epsilon V(x_{i-1})\Big)\Big]$$

we take $\epsilon=\lim_{\eta\to 0}(\epsilon-i\eta)$

$$=\int dp_i\,\exp\Big[-\frac{i\epsilon}{2m\hbar}\Big(p_i-m\frac{(x_i-x_{i-1})}{\epsilon}\Big)^2+\frac{i}{2m\hbar}m^2\frac{(x_i-x_{i-1})^2}{\epsilon^2}$$

$$-\frac{i}{\hbar}\epsilon V(x_{i-1})\Big]$$

$$=\sqrt{\frac{2\pi\hbar m}{\epsilon i}}\;\exp\Big[\frac{i}{\hbar}\Big(\frac{m}{2}\frac{(x_i-x_{i-1})^2}{\epsilon^2}-\epsilon V(x_{i-1})\Big)\Big]$$

Integrating over all of the $p(t_i)$:

$$\int \mathcal{D}p(t)\, \exp\left[\frac{i}{\hbar}\int_t^{t'} d\tau \left(p(\tau)\dot{x}(\tau) - \frac{1}{2m}p(\tau)^2 - V(x(\tau))\right)\right]$$

$$= N \exp\left[\frac{i}{\hbar}\int d\tau \left(\frac{m}{2}\dot{x}(\tau)^2 - V(x(\tau))\right)\right]$$

Normalization
- from Gaussian integrals
- indep. of x, t, x', t'

$\dot{x}(\tau) \equiv \frac{x_i - x_{i-1}}{\epsilon}$

$$\Rightarrow U(x',t';x,t) = N \int_{\substack{x(t)=x \\ x(t')=x'}} \mathcal{D}x(\tau)\, \exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau \left[\frac{\dot{x}(\tau)^2}{2m} - V(x(\tau))\right]\right\}$$

$$\equiv N \int \mathcal{D}x(\tau)\, \exp\left[\frac{i}{\hbar}\int_t^{t'} d\tau\, L\left(x(\tau), \dot{x}(\tau)\right)\right]$$

Lagrangian $L = \frac{\dot{x}^2}{2m} - V(x)$

$$= N \int \mathcal{D}x(\tau)\, \exp\left[\frac{i}{\hbar} S\left(x(\tau)\right)\right]$$

Action $S = \int d\tau\, L\left(x(\tau), \dot{x}(\tau)\right)$

- path integral representation of the propagator

A hint at classical physics:

Consider a macroscopic system such that for sample trajectories where $\frac{\delta S}{\delta x(\tau)} = 0$, $S[x(\tau)] \gg \hbar$

Then the path integral can be approximated by a stationary phase approximation and is dominated by the configuration $x(\tau)$ s.t. $\delta S = 0$.

But $\delta S = 0$ is Hamilton's principle, and leads to the Euler-Lagrange eqs. of classical mechanics.

Hence, the path integral for macroscopic systems tends to be dominated by configurations satisfying

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0}$$

This does not prove classical-like behavior, but hints at it.

Examples: For the free particle and the simple harmonic oscillator the functional integral is Gaussian and can be evaluated explicitly. The results are:

Free particle w/ $H = \frac{p^2}{2m}$:

$$U(x', t'; x, t) = \sqrt{\frac{m}{2\pi\hbar i t}}\, e^{im(x'-x)^2/2\hbar(t'-t)}$$

Simple Harmonic Oscillator w/ $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$:

$$U(x', t'; x, t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin[\omega(t'-t)]}}\, \exp\left\{\left(\frac{im\omega}{2\hbar \sin[\omega(t'-t)]}\right)\left((x'^2 + x^2)\cos[\omega(t'-t)] - 2x'x\right)\right\}$$