PHYS 621 Lecture Notes 14

N 9.1 Time-Evolution in nonconservative systems

Recall the time-evolution operator:

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle$$

$$\hat{U}(t,t_0)^\dagger = \hat{U}^{-1}(t,t_0)$$

so that $\langle\psi(t)|\psi(t)\rangle = \langle\psi(t_0)|\psi(t_0)\rangle$

<u>Conservative system</u> $\rightarrow$ $\hat{H}$ indep. of t.

$\hat{H}$ eigenstates $|\psi_E(t)\rangle \implies \hat{H}|\psi_E(t)\rangle = E|\psi_E(t)\rangle$

Completeness: $\hat{\mathbb{1}} = \sum_{n,a} |E_{n,a}\rangle\langle E_{n,a}| e^{-iE_n(t-t_0)/\hbar}$

in case of degeneracy $E_{n,a} = E_n$

$$|\psi(t)\rangle = \sum_{n,a} |E_{n,a}\rangle\langle E_{n,a}|\psi(t_0)\rangle e^{-iE(t-t_0)/\hbar}$$

$$\rightarrow \hat{U}(t,t_0) = \sum_{n,a} |E_{n,a}\rangle\langle E_{n,a}| e^{-iE(t-t_0)/\hbar}$$

$$\equiv e^{-i\hat{H}(t-t_0)/\hbar} \sum_{n,a} |E_{n,a}\rangle\langle E_{n,a}|$$

$$\hat{U}(t,t_0) = e^{-i\hat{H}(t-t_0)/\hbar}$$

$\hat{U}(t,t_0)$ satisfies
$$\begin{cases} i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}\,\hat{U}(t,t_0) \\ \hat{U}(t_0,t_0) = \mathbb{1} \end{cases}$$

Now consider non-conservative systems $\Rightarrow \hat{H}(t)$

Over short enough times such that $\hat{H}(t)$ does not change appreciably,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \implies |\psi(t+dt)\rangle = \left(\hat{1} - \frac{i}{\hbar}\hat{H}(t)dt\right)|\psi(t)\rangle$$

$$= \hat{U}(t+dt, t)|\psi(t)\rangle$$

$$= e^{-i\hat{H}(t)dt/\hbar}|\psi(t)\rangle$$

$\hat{H}(t)$ Hermitian $\hat{H}(t)^{+} = \hat{H}(t)$

$$\rightarrow \hat{U}^{+}(t+dt, t) = e^{+i\hat{H}(t)dt/\hbar} = \hat{U}^{-1}(t+dt, t)$$

Build up the evolution in small intervals

$$\boxed{\hat{U}(t,t_0) = \lim_{dt_n \to 0} \prod_{n=1}^{N} \exp\left[-\frac{i}{\hbar}\hat{H}(t_n)dt_n\right]}$$

where N is such that $\sum_{n=1}^{N} dt_n = (t-t_0)$.

In the product over N, operators involving later times are on the left.

→ Time-ordered product $\hat{U}(t,t_0) \equiv T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right]$

Time-ordered exponential, defined by the expression in the box $\boxed{}$ above.

=

Useful theorem: $\boxed{e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}\exp\left(-\frac{[\hat{A},\hat{B}]}{2}\right)}$

N9.2

Heisenberg Picture

In either the conservative or nonconservative case, define the **Schrödinger picture** states as those satisfying the Schrödinger eq: $i\hbar \frac{d}{dt}|\psi_S(t)\rangle = \hat{H}|\psi_S(t)\rangle$.

Define the corresponding Heisenberg-picture state by absorbing the time-dependence:

$$\boxed{|\psi_H\rangle \equiv \hat{U}^{-1}(t,t_0)|\psi_S(t)\rangle = |\psi_S(t_0)\rangle}$$

In the Schrödinger picture operators like $\hat{x}$ and $\hat{p}$ are time-independent.

Define time-dependent operators in the Heisenberg picture as:

$$\boxed{\hat{A}_H(t) \equiv \hat{U}^{+}(t,t_0)\,\hat{A}_S\,\hat{U}(t,t_0)}$$

Then matrix elements of Schrödinger-picture operators between Schrödinger-picture states

= matrix elements of corresponding Heisenberg-picture operators between Heisenberg-picture states

$$\boxed{\langle\psi_S(t)|\hat{A}_S|\chi_S(t)\rangle = \langle\psi_H|\hat{A}_H(t)|\chi_H\rangle}$$

Schrödinger-picture operators *may* depend explicitly on time. $\hat{A}_S(t)$

The time-dependent Heisenberg-picture operator $\hat{A}_H(t)$ satisfies

$$i\hbar \frac{d}{dt} \hat{A}_H(t) = i\hbar \frac{d}{dt}\left[\hat{U}_S^{+}(t,t_0)\,\hat{A}_S(t)\,\hat{U}_S(t,t_0)\right]$$

$$= -\hat{U}_S^{+}\,\hat{H}_S\,\hat{A}_S(t)\,\hat{U}_S + i\hbar\,\hat{U}_S^{+}\frac{\partial \hat{A}_S}{\partial t}\hat{U}_S$$

$$+ \hat{U}_S^{+}\,\hat{A}_S(t)\,\hat{H}_S\,\hat{U}_S$$

$$= \hat{U}_S^{+}\left[\hat{A}_S(t), \hat{H}_S\right]\hat{U}_S + i\hbar\,\hat{U}^{+}\frac{\partial \hat{A}_S}{\partial t}\hat{U}$$

Define the Heisenberg-picture Hamiltonian

$$\hat{H}_H(t) \equiv \hat{U}_S^{+}(t,t_0)\,\hat{H}_S(t)\,\hat{U}_S(t,t_0)$$

Then $\left[\hat{A}_H, \hat{H}_H\right] = \left[\hat{U}_S^{+}\hat{A}_S\hat{U}_S, \hat{U}_S^{+}\hat{H}_S\hat{U}_S\right]$

$$= \hat{U}_S^{+}\left[\hat{A}_S, \hat{H}_S\right]\hat{U}_S$$

If $\frac{\partial \hat{A}_S}{\partial t} \neq 0$, define $\frac{\partial \hat{A}_H}{\partial t} = \hat{U}^{+}\frac{\partial \hat{A}_S}{\partial t}\hat{U}$

Then: $$\boxed{i\hbar \frac{d\hat{A}_H}{dt} = \left[\hat{A}_H, \hat{H}_H\right] + i\hbar\frac{\partial}{\partial t}\hat{A}_H}$$

– Heisenberg Equations of Motion

Suppose $\hat{H}_S = \frac{\hat{P}_S^2}{2m} + V(\hat{x}_S)$.

$$\hat{H}_H = \frac{\hat{P}_H^2}{2m} + V(\hat{x}_H)$$

$$\hat{P}_H^2 = \hat{U}_S^\dagger \hat{P}_S \hat{U}_S \hat{U}_S^\dagger \hat{P}_S \hat{U}_S$$

$$= \hat{U}_S^\dagger \hat{P}_S^2 \hat{U}_S$$

$$[\hat{x}_H, \hat{P}_H] = \hat{U}_S^\dagger [\hat{x}_S, \hat{P}_S] \hat{U}_S = i\hbar$$

Consider $[\hat{P}_H, V(\hat{x}_H)]$.

Assume $V(\hat{x}_H)$ has a series expansion

$$V(\hat{x}_H) = \sum_n V_n \hat{x}_H^n$$

$$[\hat{p}_H, \hat{x}_H^n] = [\hat{p}_H, \hat{x}_H]\hat{x}_H^{n-1} + \hat{x}_H [\hat{p}_H, \hat{x}_H]\hat{x}_H^{n-2} + \dots + \hat{x}_H^{n-1}[\hat{p}_H, \hat{x}_H]$$

$$= -i\hbar\, n\, \hat{x}_H^{n-1}$$

$$[\hat{p}_H, V(\hat{x}_H)] = \sum_n V_n (-i\hbar\, n\, \hat{x}_H^{n-1})$$

$$= -i\hbar \frac{d}{d\hat{x}_H} V(\hat{x}_H)$$

$$\Rightarrow \boxed{[\hat{p}_H, V(\hat{x}_H)] = -i\hbar \frac{d}{d\hat{x}_H} V(\hat{x}_H)}$$

$$\frac{d\hat{x}_H}{dt} = \frac{1}{i\hbar}\left[\hat{x}_H, \left(\frac{\hat{p}_H^2}{2m} + V(\hat{x}_H)\right)\right]$$

$$= \frac{1}{i\hbar}\cdot\frac{1}{2m}\left[\hat{x}_H, \hat{p}_H^2\right]$$

$$= \frac{1}{i\hbar}\cdot\frac{1}{2m}\cdot 2i\hbar\,\hat{p}_H$$

$$\rightarrow \boxed{\frac{d\hat{x}_H}{dt} = \frac{\hat{p}_H}{m}}$$

$$\frac{d\hat{p}_H}{dt} = \frac{1}{i\hbar}\left[\hat{p}_H, \left(\frac{\hat{p}_H^2}{2m} + V(\hat{x}_H)\right)\right]$$

$$= \frac{1}{i\hbar}(-i\hbar)\frac{d}{d\hat{x}_H}V(\hat{x}_H)$$

$$\rightarrow \boxed{\frac{d\hat{p}_H}{dt} = -\frac{dV(\hat{x}_H)}{d\hat{x}_H}}$$

The Heisenberg equations of motion for $\hat{x}_H$, $\hat{p}_H$ resemble classical equations of motion.

N9.3 Example: Simple Harmonic Oscillator in Heisenberg Picture

$$i\hbar\frac{d\hat{a}}{dt} = [\hat{a}, \hat{H}] = [\hat{a}, \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)]$$

$$= \hbar\omega\,\hat{a}$$

Solution: $\boxed{\hat{a}(t) = \hat{a}_0 e^{-i\omega t}}$, $\boxed{\hat{a}^\dagger(t) = \hat{a}_0^\dagger e^{i\omega t}}$

$$\boxed{\begin{aligned}\hat{x}_H(t) &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_0^{+} e^{i\omega t} + \hat{a}_0 e^{-i\omega t}\right)\\ \hat{p}_H(t) &= i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}_0^{+} e^{i\omega t} - \hat{a}_0 e^{-i\omega t}\right)\end{aligned}}$$