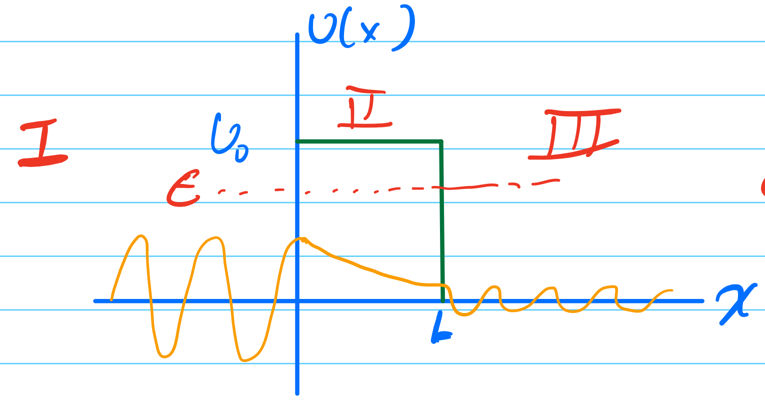
PHYS 621 Lecture Notes 12

Tunneling

Consider a potential barrier

$$U(x) = \begin{cases} 0, & x < 0 \\ U_0 > 0 & 0 < x < L \\ 0 & x > L \end{cases}$$



$\epsilon < U_0$

Region I: $x < 0$ $\psi(x) = e^{i\sqrt{\epsilon}x} + R e^{-i\sqrt{\epsilon}x}$

Region III: $x > L$ $\psi(x) = S e^{i\sqrt{\epsilon}x}$

Region II: $0 < x < L$ $\psi(x) = A e^{\kappa x} + B e^{-\kappa x}$

$$\kappa = \sqrt{U_0 - \epsilon}$$

Continuity @ $x=0$: $1 + R = A + B$

Continuity @ $x=L$: $Ae^{\kappa L} + Be^{-\kappa L} = Se^{i\sqrt{\epsilon}L}$

Continuity of $\frac{\psi'}{\psi}$ @ $x=0$: $\frac{i\sqrt{\epsilon}(1-R)}{1+R} = \frac{\kappa(A-B)}{A+B}$

Cont. of $\frac{\psi'}{\psi}$ @ $x=L$: $\frac{\kappa(Ae^{\kappa L} - Be^{-\kappa L})}{Ae^{\kappa L} + Be^{-\kappa L}} = i\sqrt{\epsilon}$

Solve continuity equations:

(1) $R = A + B - 1$

(2) $S = e^{-i\sqrt{E}L}\left(Ae^{\kappa L} + Be^{-\kappa L}\right)$

(3) $i\sqrt{E}\,\dfrac{(1-R)}{1+R} = \dfrac{\kappa(A-B)}{A+B}$

$= i\sqrt{E}\,\dfrac{(2-A-B)}{A+B}$

$A - B = \dfrac{i\sqrt{E}}{\kappa}(2-A-B)$

(4) $\left(Ae^{\kappa L} - Be^{-\kappa L}\right) = \dfrac{i\sqrt{E}}{\kappa}\left(Ae^{\kappa L} + Be^{-\kappa L}\right)$

- Solve (3) and (4) for A, B
- Substitute in (1) and (2)

$\Rightarrow$ Transmission coefficient

$$T = |S|^2 = \left(1 + \frac{V_0^2 \sinh^2(\kappa L)}{4E(V_0 - E)}\right)^{-1}$$

$\dfrac{k_{III}}{k_I} = \dfrac{\sqrt{E}}{\sqrt{E}} = 1$

Reflection coefficient

$$|R|^2 = 1 - T = \frac{V_0^2 \sinh^2(\kappa L)}{4E(V_0 - E) + V_0^2 \sinh^2 \kappa L}$$

The transmission coefficient $T < 1$ describes the probability that a wavepacket composed of modes localized around $E < V_0$. Classically, a particle cannot pass through a region with $E < V_0$, but quantum mechanically a particle <u>can</u>.

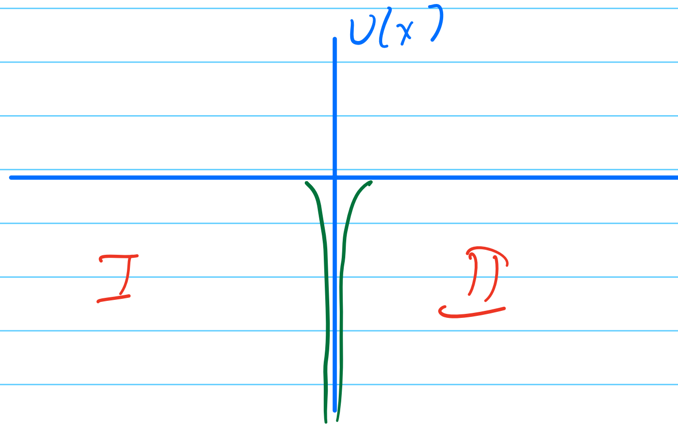
— <u>Tunneling</u>

Note that as the classically unallowed region extends further, i.e. as L increases, T decreases exponentially $\sim e^{-2\kappa L}$

Delta-Function potential

Deep potentials are sometimes modeled as delta functions.

Consider $U(x) = -U_0\,\delta(x)$



There are continuum solutions with $\epsilon > 0$, and bound state solutions w/ $\epsilon < 0$.

Bound state spectrum: $\epsilon < 0$

Region I, $x<0$ $\qquad \psi(x) = A e^{\kappa x}$ $\qquad \kappa = \sqrt{\epsilon}$

Region II, $x>0$ $\qquad \psi(x) = B e^{-\kappa x}$

Continuity at $x=0$: $A = B$

$$\rightarrow \psi(x) = A e^{-\kappa |x|}$$

The derivative of $\psi(x)$ is not continuous across the delta function!

$$\psi'(x)\Big|_{x\to 0^+} = -\kappa A$$

↑ x approaches 0 from above (+)

$$\psi'(x)\Big|_{x\to 0^-} = +\kappa A$$

$$\boxed{\Delta\psi'(x)\Big|_{x=0} = -2\kappa A}$$

Schrödinger eq:

$$-\psi''(x) - V_0\,\delta(x)\,\psi(x) = \epsilon\,\psi(x)$$

$$\int_{-\varepsilon}^{\varepsilon} dx\left(-\frac{d^2\psi}{dx^2} - V_0\,\delta(x)\,\psi(x)\right) = \cancel{\int_{-\varepsilon}^{\varepsilon} dx\,\epsilon\,\psi(x)}$$

→ 0 as $\varepsilon \to 0$

$\varepsilon \to 0$:

$$-\Delta\psi'(x)\Big|_{x=0} - V_0\,\psi(0) = 0$$

$$\rightarrow\ 2\kappa = V_0 \ \longrightarrow\ \boxed{\epsilon = \kappa^2 = \frac{1}{4}V_0^2}$$

Continuity Equation

If we normalize the wavefunction such that $\int dx\, |\psi(x, t=0)|^2 = 1$, then it's important that Schrödinger's Eq. is consistent w/ $\int dx\, |\psi(x,t)|^2 = 1$ at all times

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi$$

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + V(x)\psi^*$$

$$\Rightarrow i\hbar \left[\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}\right] = i\hbar \frac{\partial}{\partial t} |\psi|^2$$

$$= -\frac{\hbar^2}{2m} \left[\psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2}\right]$$

$$= -\frac{\hbar^2}{2m} \frac{\partial}{\partial x} \left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}|\psi|^2 = -\frac{\hbar}{2mi}\frac{\partial}{\partial x}\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right]$$

or,
$$\boxed{\frac{\partial}{\partial t}|\psi|^2 + \frac{\partial}{\partial x}J(x,t) = 0}$$

Continuity Equation

where

$$\boxed{J(x,t) = \frac{\hbar}{2mi}\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right]}$$

Integrating over x:

$$\int_{-\infty}^{\infty} dx\, \frac{\partial}{\partial t}|\psi|^2 + \int_{-\infty}^{\infty} dx\, \frac{\partial}{\partial x}J(x,t) = 0$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\, |\psi|^2 + J(x,t)\Big|_{x=-\infty}^{+\infty} = 0$$

Assuming $\psi(x,t)$ approaches 0 @ $\pm\infty$,

$J(x,t) \to 0$ as $x \to \pm\infty$.

Then,
$$\boxed{\frac{d}{dt}\int_{-\infty}^{+\infty} dx\, |\psi(x,t)|^2 = 0}$$

Hence, if $\Psi(x, t=0)$ is normalized, so is $\Psi(x,t)$ $\forall t$.

This allows for the interpretation of $|\Psi(x,t)|^2$ as a probability density.