PHYS 621 Fall '24 Quantum Mechanics

This is the first semester of a two-semester graduate course on quantum mechanics. Most of the course will focus on <u>nonrelativistic</u> quantum mechanics.

Our goals are to develop a deeper understanding of the fundamentals of quantum theory and to develop techniques for applying quantum theory to more complex systems than you may have encountered in an undergraduate setting!

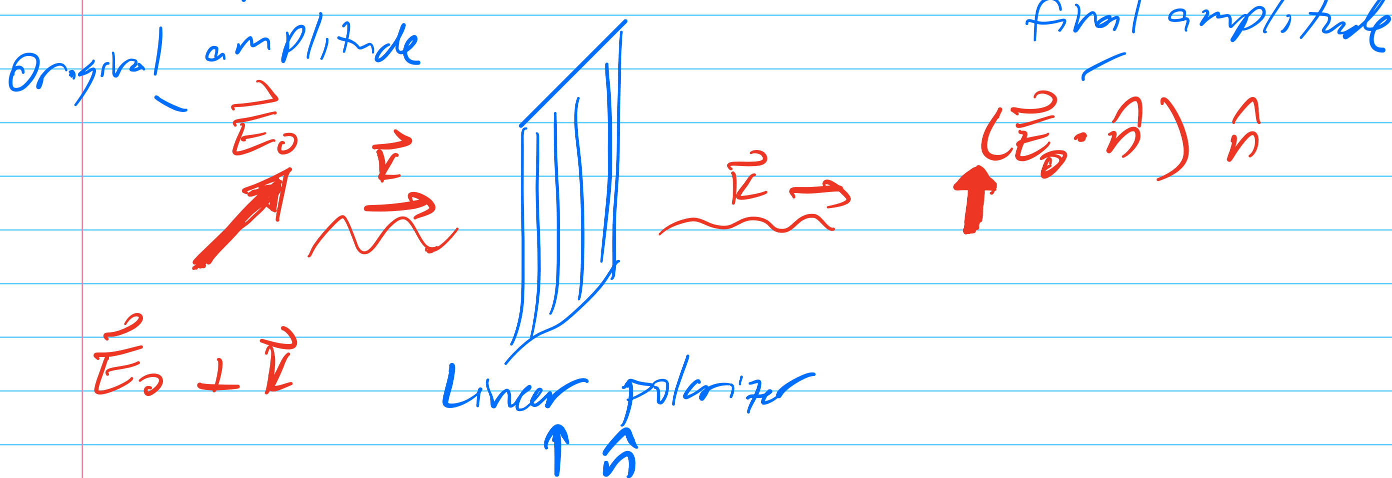
Although photons, quanta of light, are massless → relativistic, they are useful for illustrating some important features of quantum mechanics.

1) <u>Classical electromagnetic plane waves</u>

$$\vec{E} = \vec{E}_0 \cos(\omega t - \vec{k}\cdot\vec{x} - \phi)$$

some phase

Energy $\sim |\vec{E}|^2$



After passing through a linear polarizer,
unless the electric field originally points along the polarizer axis, the amplitude of the wave is reduced after passing through the polarizer.

Energy density originally $U_i = \epsilon_0 E_0^2$

Energy density after polarizer $U_f = \epsilon_0 (\vec{E}_0 \cdot \hat{n})^2$
$\leq U_i$

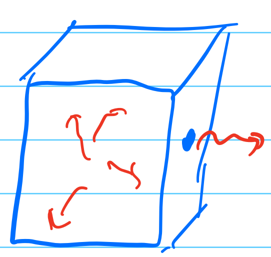
Some energy is dissipated in the polarizer.

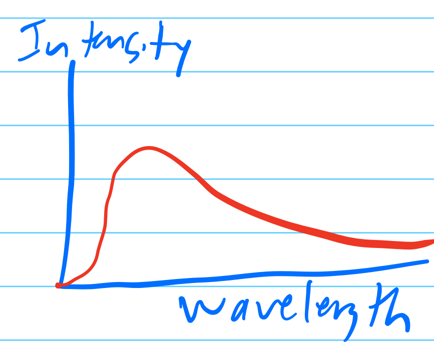
2) Photons

- In order to explain the spectrum of blackbody radiation, in 1900, Max Planck suggested that energy could only be transferred between matter and electromagnetic radiation in integer units of quantized energy

$$\Delta E = n h \nu \quad , \quad \nu \lambda = c$$

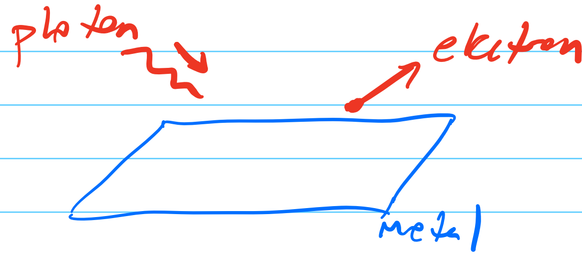
h: Planck's const. ν: frequency λ: wavelength





- In 1905 Einstein explained the photoelectric effect by postulating that light is composed of quanta with

$$\boxed{E = h\nu}$$



$$E_{kin} = \frac{mv^2}{2} = h\nu - W$$

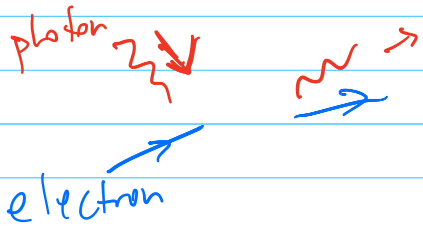
work function of the metal

As the amplitude of the electromagnetic wave increases, the energy of emitted electrons stays the same.

- Compton effect (1923) → photons have momentum

$$\boxed{p = \frac{h\nu}{c} = \frac{h}{\lambda}}$$

scattering ∡



$$\Delta\lambda = \frac{2h}{mc}\sin^2\frac{\theta}{2}$$

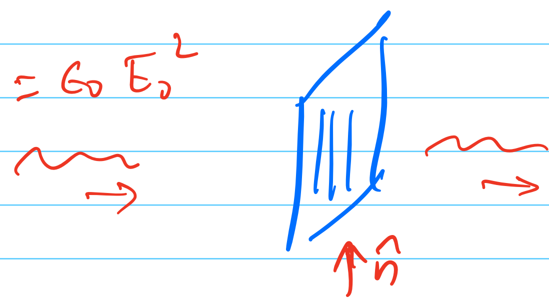
electron mass

Reconciling the classical and quantum pictures required a dramatic rethinking of the fundamental descriptions of physics.

$\boxed{\text{Indeterminism}}$

Let's think about the polarizer.

$U_i = \epsilon_0 E_0^2$



$U_f = \epsilon_0 (\vec{E}_0 \cdot \hat{n})^2$

If light is composed of photons, how does the photon lose energy when passing through the polarizer?

Answer: A fraction of the photons passes through the polarizer.

The other ones don't.

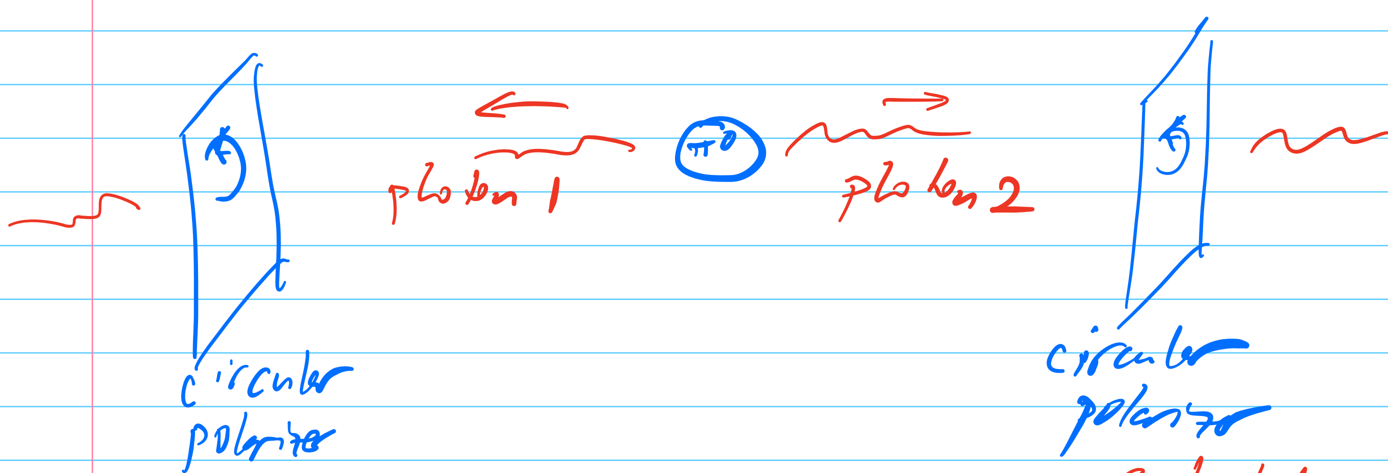
How do we know whether a photon will go through? If the photons are identical?

$\longrightarrow$ probabilities

A natural perspective at this point is that perhaps it's naive to think that actual photons are identical. They will have different momenta, will hit the polarizer in different places, etc.

Maybe this detailed information, or "hidden variables" determine whether a photon gets through the polarizer or not.

Well...



The helicities of photons can be correlated, so (entangled) if physics is local, it can't be where the photon hits the screen that matters.

If we had more info about the photon could we in principle predict whether it will get through the polarizer?

— Bell's Inequality violation → entangled particles can have correlated observables that can not be explained by local hidden variables.

→ Can't get away without probabilities or nonlocality

To describe the state of the photon, we can use Dirac's notation.

$$|\text{photon}, \vec{k}, \hat{\varepsilon}\rangle$$

wavevector polarization axis

$$\hat{\varepsilon} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$ — vector addition

Before

$$|\text{photon}, \vec{k}, \hat{\varepsilon}\rangle = \cos\theta\,|\text{photon}, \vec{k}, \hat{x}\rangle + \sin\theta\,|\text{photon}, \vec{k}, \hat{y}\rangle$$

Suppose the polarizer axis is $\hat{n} = \hat{y}$.

After passing through the polarizer the state is

After

$$|\text{photon}, \vec{k}, \hat{y}\rangle$$

Probability of photon getting through must be

$$\text{prob} = (\hat{\varepsilon}\cdot\hat{y})^2 = \sin^2\theta$$

to match the change in energy density after passing through the polarizer.

Superposition and state normalization

The decomposition of the state $|\text{photon}, \vec{k}, \hat{\varepsilon}\rangle$ into a $|\text{photon}, \vec{k}, \hat{x}\rangle$ component and a $|\text{photon}, \vec{k}, \hat{y}\rangle$ component is a kind of superposition.

The polarizer "projected out" one of the two polarizations.

$$|\text{photon}, \vec{k}, \hat{\varepsilon}\rangle = \cos\theta\, |\text{photon}, \vec{k}, \hat{x}\rangle + \sin\theta\, |\text{photon}, \vec{k}, \hat{y}\rangle$$

$$\xrightarrow{\text{polarizer}} \cancel{\cos\theta\, |\text{photon}, \vec{k}, \hat{x}\rangle} + \sin\theta\, |\text{photon}, \vec{k}, \hat{y}\rangle$$

What does the $\sin\theta$ mean after the photon passes the polarizer?

Answer: Nothing.

The overall normalization of a state is just a convention.

Nakahara ch. 1

Mathematical digression: Finite-dimensional Vector Spaces

A vector space $\mathbb{V}$ is a collection of elements denoted $|v\rangle$ in Dirac's 'ket' notation, which can be multiplied by a scalar and added together (c.f. superposition)

- $\forall\ |v\rangle, |w\rangle \in \mathbb{V}$, $|v\rangle + |w\rangle \in \mathbb{V}$
- $\forall\ \alpha \in \mathbb{C}$ and $|v\rangle \in \mathbb{V}$, $\alpha|v\rangle \in \mathbb{V}$.

complex #s

- Distributive law: $\forall\ \alpha, \beta \in \mathbb{C}$ and $|v\rangle \in \mathbb{V}$,

$$(\alpha+\beta)|v\rangle = \alpha|v\rangle + \beta|v\rangle$$

Also, $\forall\ \alpha \in \mathbb{C}$, $|v\rangle, |w\rangle \in \mathbb{V}$,

$$\alpha(|v\rangle + |w\rangle) = \alpha|v\rangle + \alpha|w\rangle$$

- Addition of vectors and multiplication by scalars is commutative, e.g.

$$|v\rangle + |w\rangle = |w\rangle + |v\rangle,$$

and associative e.g.

$$(|v\rangle + |w\rangle) + |u\rangle = |v\rangle + (|w\rangle + |u\rangle)$$

- $\exists$ unique zero vector $|0\rangle$ s.t. $|v\rangle + |0\rangle = |v\rangle$ $\forall\, |v\rangle \in \mathbb{V}$.

- $\exists$ unique inverse under addition:

  $\forall\, |v\rangle$, $\exists\, |-v\rangle$ s.t. $|v\rangle + |-v\rangle = |0\rangle$.

Vectors $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$ are <u>linearly dependent</u> if there is a linear combination of them that vanishes, i.e.

$\exists\, \alpha_i$, not all zero, s.t. $\sum_{i=1}^{n} \alpha_i |v_i\rangle = |0\rangle$.

A vector space is n-dimensional if there is a set of n linearly independent vectors $|v_1\rangle, \ldots, |v_n\rangle$, but all sets of $n+1$ vectors are linearly dependent.

The set $\{|v_1\rangle, \ldots, |v_n\rangle\}$ is a <u>basis</u> for the vector space.

Then <u>any</u> $|v\rangle \in \mathbb{V}$ can be decomposed as

$$|v\rangle = \sum_{i=1}^{n} \alpha_i |v_i\rangle$$

The choice of α_i is unique given $|v\rangle$.

Proof: Assume the opposite. Then $\exists \{\tilde{\alpha}_i\}$ s.t. $|v\rangle = \sum_{i=1}^{n} \tilde{\alpha}_i |v_i\rangle$ and

$$|v\rangle = \sum_{i=1}^{n} \alpha_i |v_i\rangle$$

Then taking the difference,

$$|0\rangle = \sum_{i=1}^{n} (\tilde{\alpha}_i - \alpha_i) |v_i\rangle$$

Therefore, $\{|v_i\rangle\}$ are linearly dependent, contrary to our premise. $\boxtimes$

Subspaces

If there is a subset $\mathbb{V}' \subset \mathbb{V}$ such that $\forall |v\rangle \in \mathbb{V}'$ and $\forall \alpha \in \mathbb{C}$ then $\alpha|v\rangle \in \mathbb{V}'$ and also $\forall |v\rangle, |w\rangle \in \mathbb{V}'$ then $|v\rangle + |w\rangle \in \mathbb{V}'$, then $\mathbb{V}'$ is a subspace of $\mathbb{V}$, with dimension $k \leq n$.

Scalar Product, orthonormality

Given two vectors $|v\rangle, |w\rangle$, their scalar product is a complex number

$\langle w|v\rangle$, with the properties:

- $\langle w|v\rangle = \langle v|w\rangle^*$
- $\langle w|(\alpha|v\rangle + \beta|u\rangle) = \alpha\langle w|v\rangle + \beta\langle w|u\rangle$
- $\langle v|v\rangle \geq 0$, and iff $\langle v|v\rangle = 0 \Leftrightarrow |v\rangle = |0\rangle$.
- $\langle \alpha v + \beta u|w\rangle = \alpha^*\langle v|w\rangle + \beta^*\langle u|w\rangle$

Notation indicating the inner product of $|w\rangle$ and $\alpha|v\rangle + \beta|u\rangle = |\alpha v + \beta u\rangle$.

Two vectors $|v\rangle, |w\rangle$ are <u>orthogonal</u> iff $\langle v|w\rangle = 0$.

Note that $\langle v|0\rangle = \langle v|w-w\rangle$

$$= \langle v|w\rangle - \langle v|w\rangle$$

$$= 0.$$

<u>Norm</u> of a vector is

$$\boxed{\|v\| \equiv \sqrt{\langle v|v\rangle}}$$

Two vectors $|v\rangle, |w\rangle$ are orthonormal if $\langle v|w\rangle = 0$ and $\|v\| = \|w\| = 1$.

Given a basis of orthonormal vectors $\{|i\rangle\}$, any vector $|v\rangle$ can be decomposed as

$$|v\rangle = \sum_i v_i |i\rangle$$

Suppose also $|w\rangle = \sum_i w_i |i\rangle$

Then
$$\langle v|w\rangle \sum_{i,j} v_i^* w_j \langle i|j\rangle$$
$$= \sum_{i,j} v_i^* w_j \, \delta_{ij}$$

"kronecker delta" $= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$$\boxed{\langle v|w\rangle = \sum_i v_i^* w_i}$$

Norm $\|v\|^2 = \langle v|v\rangle = \sum_i |v_i|^2$.

If $|j\rangle$ is one of the orthonormal basis vectors, then

$$\langle j|v\rangle = \sum_i v_i \langle j|i\rangle = \sum_i v_i \delta_{ij}$$

$$\boxed{\langle j|v\rangle = v_j}$$

Then $|v\rangle = \sum_i |i\rangle \underbrace{\langle i|v\rangle}_{v_i}$

Gram - Schmidt theorem

Given any linearly independent basis, we can always find an orthonormal basis by making linear combinations of the basis vectors.

N 1.2

Operators on a Vector Space

An operator $\hat{A}$ is an action on a vector space back into the same vector space

$$\hat{A} : |v\rangle \to |w\rangle$$

Linear operators:

$$\hat{A}(\alpha |v\rangle + \beta |w\rangle) = \alpha \hat{A} |v\rangle + \beta \hat{A} |w\rangle$$

Antilinear operators:

$$\hat{A}(\alpha |v\rangle + \beta |w\rangle) = \alpha^* \hat{A} |v\rangle + \beta^* \hat{A} |w\rangle$$

Neutral operator under addition $\hat{0}$

$$\hat{0}|v\rangle = |0\rangle \qquad \forall\ |v\rangle \in V$$

Neutral operator under multiplication $\hat{1}$

$$\hat{1}|v\rangle = |v\rangle \qquad \forall\ |v\rangle \in V$$

$$(\alpha \hat{A} + \beta \hat{B})|v\rangle = \alpha (\hat{A}|v\rangle) + \beta (\hat{B}|v\rangle)$$

Product of operators = Composition of operators
(act with one first, then the other
— order matters!)

$$\hat{A}\hat{B}|v\rangle = \hat{A}(\hat{B}|v\rangle)$$

Completeness Relation:

$$|v\rangle = \sum_i |i\rangle\langle i|v\rangle$$

$$\rightarrow \sum_i |i\rangle\langle i| = \hat{\mathbb{1}}$$

More generally, we can write

$$\hat{A} = \sum_{\substack{a\in A\\ b\in B}} |a\rangle\langle b|$$ ← A, B = some set of vectors

$$\hat{A}|v\rangle = \sum_{\substack{a\in A\\ b\in B}} |a\rangle\langle b|v\rangle$$

$$= \sum_{\substack{a\in A\\ b\in B}} |a\rangle v_b \in \mathbb{V}.$$

→ Matrix Representation

$$|v\rangle = \sum_i |i\rangle v_i = \sum_i |i\rangle\langle i|v\rangle$$

basis vector coefficient

→ similar to $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

coefficient basis vector

— Represent $|v\rangle$ as a column vector

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \langle 1|v_1\rangle \\ \langle 2|v_2\rangle \\ \vdots \\ \langle n|v_n\rangle \end{pmatrix}$$