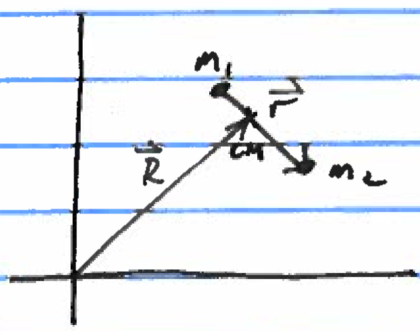
The Central Force Problem - Goldstein, Ch. 3

We will consider the problem of two bodies moving under the influence of a mutual central force.

We can describe the motion in terms of the displacement of the center of mass from the origin, $\vec{R}$, and the displacement of object 2 with respect to object 1, $\vec{r}_2 - \vec{r}_1 \equiv \vec{r}$



$$L = T(\dot{\vec{R}}, \dot{\vec{r}}) - U(\vec{r}, \dot{\vec{r}})$$

The displacement of m_1 from the Center of Mass (CM) is

$$\vec{r}_1{}' = -\frac{m_2}{m_1+m_2}\,\vec{r}$$

Similarly, the displacement of m_2 from the CM is,

$$\vec{r}_2{}' = \frac{m_1}{m_1+m_2}\,\vec{r}.$$

Check: $\dfrac{m_1\vec{r}_1{}' + m_2\vec{r}_2{}'}{m_1+m_2} = \vec{r}_{CM}{}' = \vec{0}.$ ← Displacement of CM from CM.

The kinetic energy is a sum of terms from the motion of the CM and the motion about the CM:

$$T = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}m_1(\dot{\vec{r}}_1')^2 + \frac{1}{2}m_2(\dot{\vec{r}}_2')^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}m_1\left(\frac{-m_2}{m_1+m_2}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\dot{\vec{r}}\right)^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2$$

Reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

$$L = T - U = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r},\dot{\vec{r}})$$

The center of mass coordinates in $\vec{R}$ are cyclic, so $\vec{P}_{cm} = (m_1+m_2)\dot{\vec{R}}$ is conserved.

The motion $\vec{r}(t)$ about the center of mass is decoupled from the motion of the center of mass, so from now on we focus on that motion, governed by Lagrangian

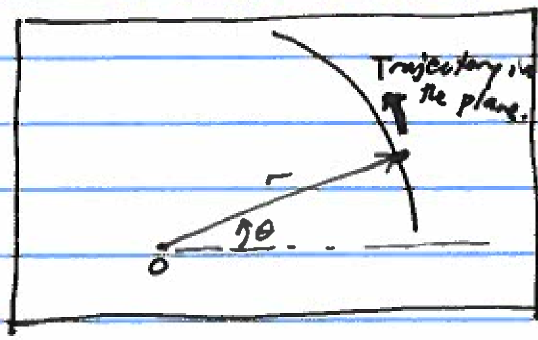
$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r},\dot{\vec{r}}).$$

Assume $U(\vec{r}_1, \vec{r}_2) = V(r)$, where $\vec{r} = |\vec{r}|$.
Then the force $\vec{F} = -\nabla V$ is along $\vec{r}$.

By spherical symmetry, $\vec{L} = \vec{r} \times \vec{p}$ is conserved.

$\vec{L}$ is perpendicular to both $\vec{r}$ and $\vec{p}$, and is constant.

→ The motion is in the plane $\perp \vec{L}$.

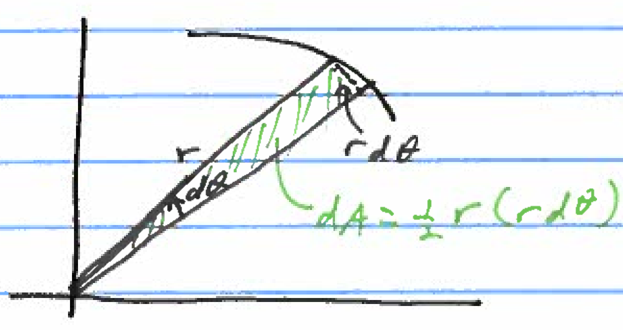


$$L = T - V = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

θ is cyclic → $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$ is conserved

= magnitude of the angular momentum

$\equiv \ell$.



$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{\ell}{2\mu}$$

= conserved

— Kepler's second law of planetary motion

Lagrange Eqn for r:

$$\frac{d}{dt}(\mu\dot{r}) - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$\mu\ddot{r} - \mu r\dot{\theta}^2 = -\frac{\partial V}{\partial r} \equiv f(r) \quad \text{force along } r$$

$$\boxed{\mu\ddot{r} - \frac{\ell^2}{\mu r^3} = f(r)} \quad \text{Independent of } \theta.$$

L does not depend explicitly on time → Energy E is conserved

$$\boxed{E = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)} \quad \text{conserved}$$

From before, $\boxed{\ell = \mu r^2\dot{\theta}}$ conserved

E and ℓ represent two integrals of the motion.

We can write $E = \frac{1}{2}\mu\left(\dot{r}^2 + \frac{\ell^2}{\mu^2 r^2}\right) + V(r)$,

or,

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - V(r) - \frac{\ell^2}{2\mu r^2}\right)}$$

We can find t as a function of r: Suppose $t=0$ when $r=r_0$.

$$t = \int dt = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V(r) - \frac{\ell^2}{2\mu r^2}\right)}}$$

Inverting this would give $r(t)$.

we could then find $\theta(t)$ to complete the solution for the motion:

$$\theta = \int d\theta = \int_0^t \frac{\ell}{\mu r^2(t)}\,dt + \theta_0, \text{ where } \theta=\theta_0 \text{ when } t=0.$$

Classification of Orbits

Suppose we want to analyze the motion with given E and ℓ.

$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{1}{2}\frac{\ell^2}{\mu r^2} + V(r)}_{V_{eff}(r)}$$

This is analogous to one-dimensional motion in a potential

$$\boxed{V_{eff}(r) = \frac{\ell^2}{2\mu r^2} + V(r)}$$

centrifugal term.

Effective force:

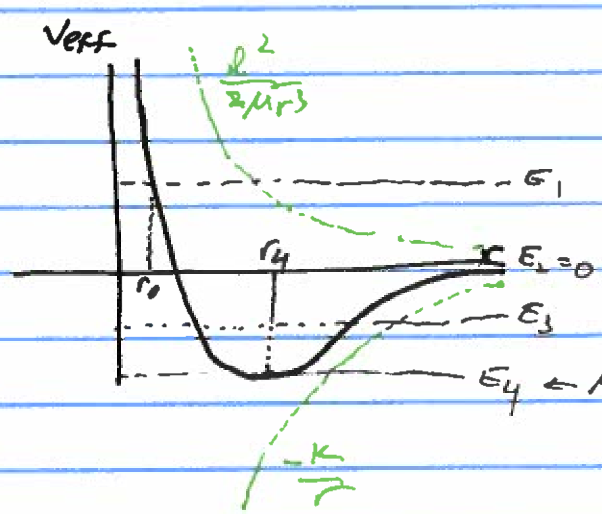
$$f_{eff} = \frac{-\partial V_{eff}}{\partial r} = \frac{\ell^2}{\mu r^3} - \frac{\partial V}{\partial r}$$

$$\frac{\ell^2}{\mu r^3} = \mu r\dot{\theta}^2 = \mu\frac{v_\theta^2}{r}$$

Example: Attractive inverse-square force

$$f = -\frac{k}{r^2}, \qquad V = -\frac{k}{r}$$

$$V_{eff} = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2}$$



Kinetic energy > 0 → Energy $> \min(V_{eff}) = E_4$ in the plot above.

↳ Also, $V(r) < E$ along the motion.

E_1: For the system with energy E_1, r remains $>$ the value where $V_{eff} = E_1$, call it r_0.

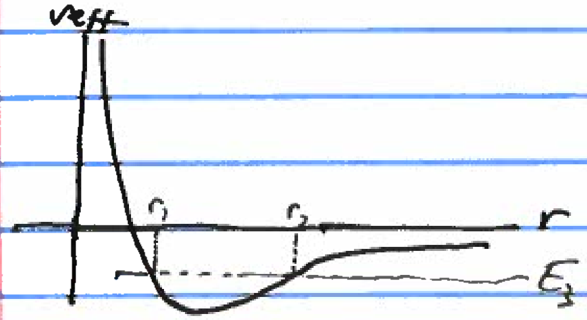
$$E - V_{eff}(r) = \frac{1}{2} m \dot{r}^2 \geq 0$$

If $\dot{r} < 0$ initially, then $r(t)$ approaches r_0, the turning point, after which $\dot{r} > 0$ and $r(t)$ recedes from r_0.

The orbit is unbound.

E_2: Similar to E_1, but $\dot{r} \to 0$ as $r \to \infty$.

E_3:



$E_3 > V_{eff}(r)$ during the motion, which is therefore bound between the turning points r_1 and r_2 where $V_{eff} = E_3$.

Note: we have not shown that the bounded orbits are closed.

E_4: E_4 is the minimum of $V_{eff}(r)$, $r \equiv r_4$.

The effective force vanishes at r_4:

$$f_{eff} = f(r) + \frac{\ell^2}{\mu r^3} = -\frac{\partial V_{eff}}{\partial r} \Rightarrow 0 \text{ at } r = r_4$$

$$\rightarrow f(r_4) = -\frac{\ell^2}{\mu r_4^3} = -\mu \frac{v_\theta^2}{r}, \text{ where } v_\theta = r\dot{\theta}$$

→ The attractive force supplies the centripetal force required for the circular motion.

Example: Motion about a Schwarzschild black hole.

In general relativity, the analogy to Noether's theorem is that there is a conserved quantity for every transformation of the coordinates (continuous and connected to the identity) that leaves the spacetime metric invariant.

The Schwarzschild black hole spacetime is spherically symmetric and time-translation invariant.

→ There is a conserved (analogy to) angular momentum, $\tilde{\ell}$, and energy, $\tilde{E}$, per unit mass of the moving object.

An analysis of the motion leads to the following equation for motion in terms of the radial coordinate r:

$$\boxed{\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(1-\frac{2GM}{r}\right)\left(k+\frac{\tilde{\ell}^2}{r^2}\right) = \frac{1}{2}\tilde{E}^2}$$

(speed of light $c=1$)

$k = \begin{cases} +1 & \text{massive particle} \\ 0 & \text{massless particle (photon)} \end{cases}$

per unit mass

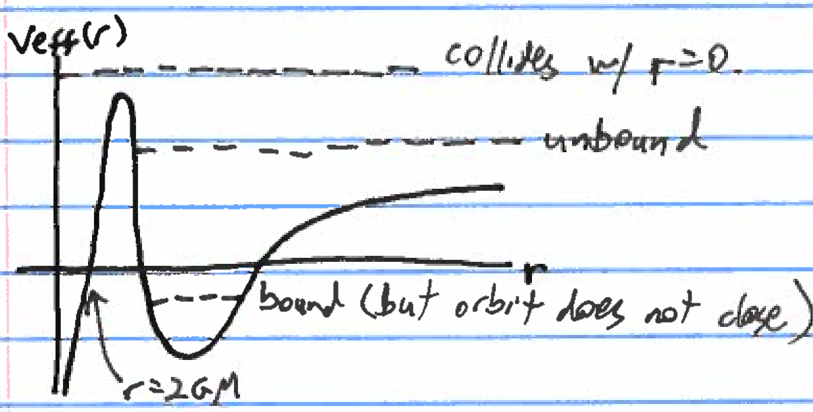
$$V_{eff}(r) = \frac{1}{2}k - \frac{kGM}{r} + \frac{\tilde{\ell}^2}{2r^2} - \frac{GM}{r^3}\tilde{\ell}^2$$

For massive particle motion, $k=+1$,

$$V_{eff}(r) = \underbrace{\frac{1}{2}}_{\text{irrelevant constant}} - \underbrace{\frac{GM}{r}}_{\text{gravitational potential}} + \underbrace{\frac{\tilde{\ell}^2}{2r^2}}_{\text{centrifugal term}} - \underbrace{\boxed{\frac{GM}{r^3}\tilde{\ell}^2}}_{\text{Non-Newtonian term.}}$$

- dominates at small $r \ll GM$
- irrelevant at $r \gg GM$.

$\kappa=+1$:

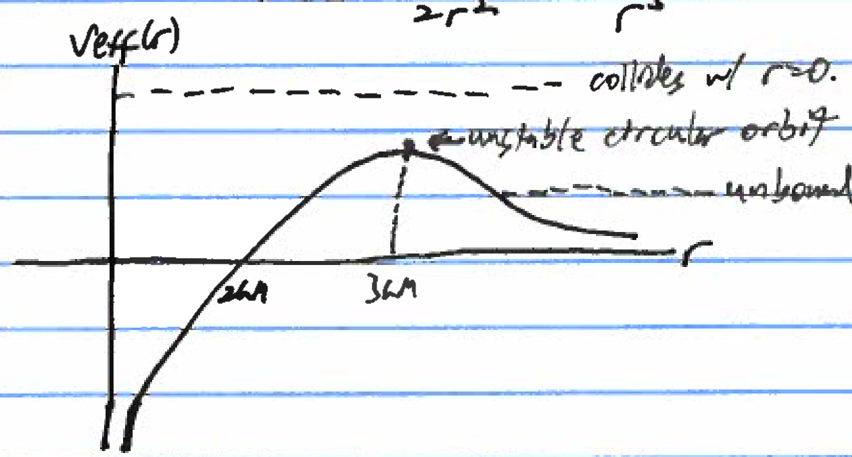


Trajectories with enough energy (for a given $\tilde{\ell}$) collide with the singularity at $r=0$.

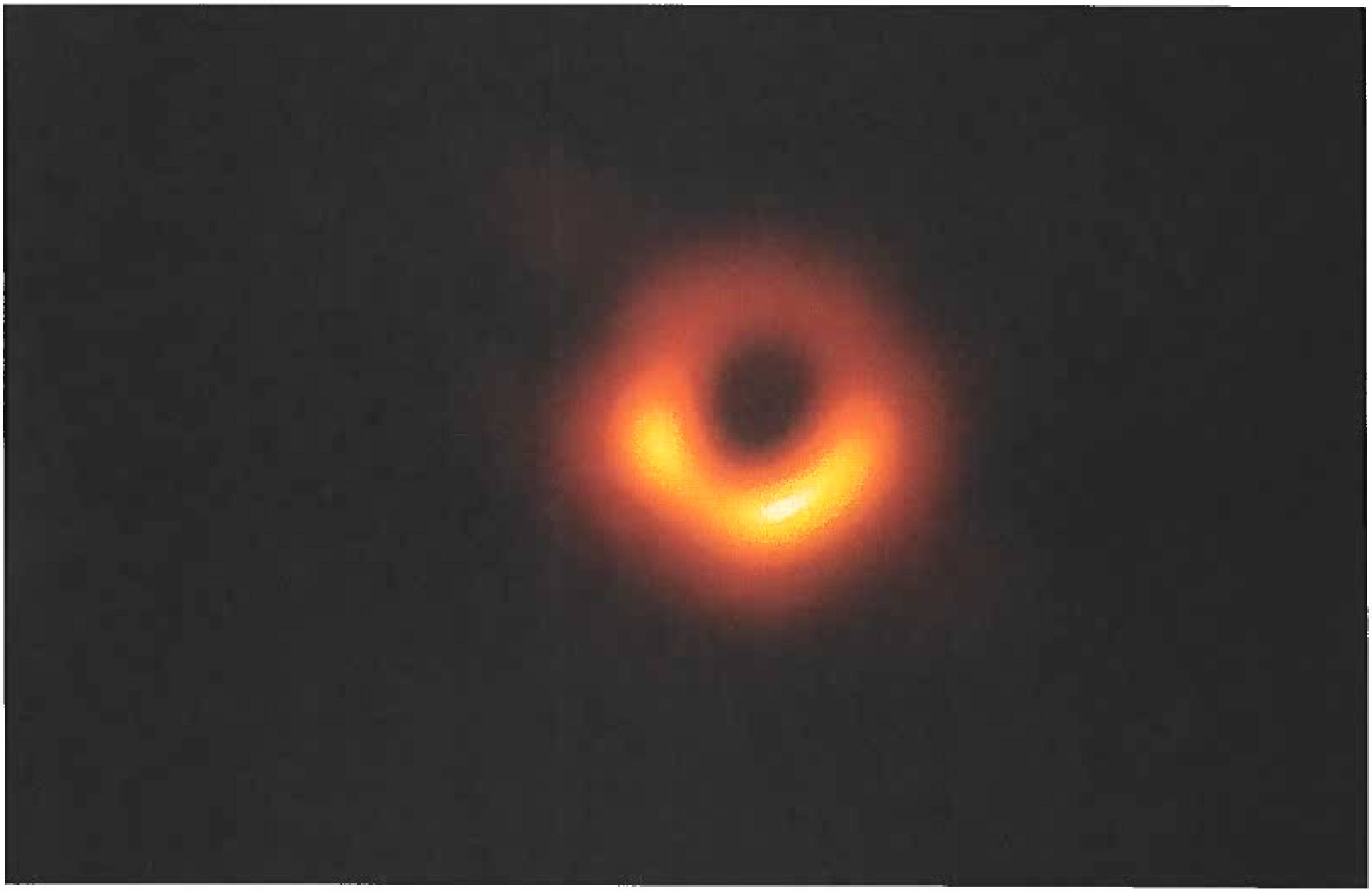
Light moves in the effective potential with $\kappa=0$:

$$V_{eff}(r) = \frac{\tilde{\ell}^2}{2r^2} - \frac{GM}{r^3}\tilde{\ell}^2$$

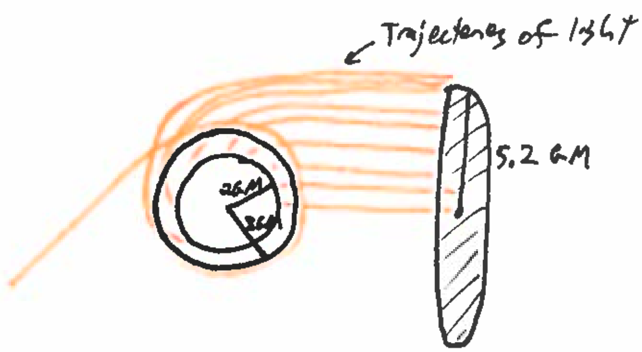
$\kappa=0$:



The only closed orbit of light is the circular orbit at $r=3GM$.



Event Horizon Telescope - Image of 6.5 billion solar mass black hole in center of M87. $\lambda = 1.3$ mm.
April 10, 2019.



Size of black hole shadow.

$r = 2GM$ Schwarzschild radius
$r = 3GM$ photon sphere
$r = 5.2GM$ Black hole shadow