More Examples of the Use of Variational Principles

Recap: To find the equations of motion for systems with (generalized) holonomic constraints:

1. Identify a set of generalized coordinates. The coordinates should either be independent or connected by equations of constraint.

↓

2. Identify constraints as equations for generalized coordinates (holonomic) or their time-derivatives (semi-holonomic).

Can you solve the constraint equations analytically?

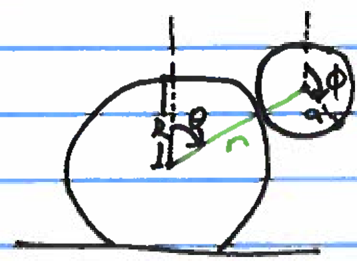
No → Use Lagrange Multipliers

Yes → Do you want to know the constraint forces?

- Yes → Use Lagrange Multipliers
- No → Use constraints to write the Lagrangian in terms of indep. generalized coords.

Example: Hoop of radius a rolling down a cylinder of radius R, from rest at the top of the cylinder. Where does the hoop fall off the cylinder?

Strategy: The hoop falls off the cylinder where the normal force vanishes. The normal force is related to the constraint fixing the distance of the hoop's center of mass (CM) to the center of the cylinder, $r = R + a$. So, we introduce a Lagrange multiplier for that constraint.



Constraints:

$f_1 = r - (R+a) = 0$ Hoop is on cylinder.

$f_2 = (R+a)\dot{\theta} - a\dot{\phi} = 0$

— Velocity of hoop at point of contact vanishes → speed of center of mass = speed of rotation of pt. on the hoop.

Kinetic Energy: $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}ma^2\dot{\phi}^2$

Motion of CM — Motion about CM

Potential Energy: $V = mgr\cos\theta$

The semi-holonomic constraint $f_2 = 0$ is related to the friction force responsible to rolling. We do not need to know this force, so we do not need a corresponding Lagrange multiplier. Instead, use $\dot{\phi} = \left(\frac{R+a}{a}\right)\dot{\theta}$.

Action with Lagrange multiplier for $f_1 = 0$:

$$I = \int_{t_1}^{t_2} dt \left\{ \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} m a^2 \left(\frac{R+a}{a}\right)^2 \dot{\theta}^2 - mgr\cos\theta + \lambda(r - (R+a)) \right\}$$

Lagrange Eqs:

$$\delta r: \quad \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + mg\cos\theta - \lambda = 0$$

$$\delta\theta: \quad \frac{d}{dt}\left(mr^2\dot{\theta} + m(R+a)^2\dot{\theta}\right) - mgr\sin\theta = 0$$

constraint: $r = R+a \rightarrow \dot{r} = \ddot{r} = 0$.

Using the constraint in the θ Eqn. gives:

$$\frac{d}{dt}\left(2m(R+a)^2\dot{\theta}\right) - mg(R+a)\sin\theta = 0$$

$$\ddot{\theta} = \frac{g}{2(R+a)}\sin\theta$$

Solution with $\dot{\theta} = 0$ when $\theta = 0$:

$$\boxed{\dot{\theta}^2 = \frac{g}{R+a} - \frac{g}{R+a}\cos\theta}$$

Substituting in the r Eqn. gives:

$$\lambda = -m(R+a)\left(\frac{g}{R+a} - \frac{g}{R+a}\cos\theta\right) + mg\cos\theta$$

$$\boxed{\lambda = 2mg\cos\theta - mg.}$$

The generalized force for r is $\vec{F} \cdot \frac{\partial \vec{r}}{\partial r} = F_r$.
This is the normal force constraining the loop to the cylinder.
In terms of the Lagrange multiplier λ:

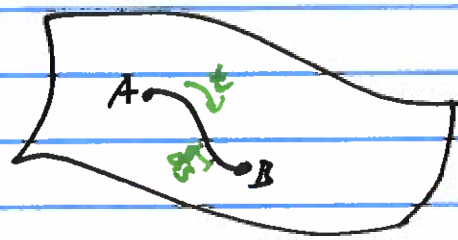
$$F_r = \lambda \frac{\partial f_1}{\partial r} = \lambda.$$

The loop leaves the cylinder when λ ceases to be positive, and the constraint $f_1 = 0$ is no longer valid:

$$\lambda = mg(2\cos\theta - 1) = 0$$

$$\rightarrow \boxed{\cos\theta = 1/2.}$$ This is where the loop falls off the cylinder.

Example: Geodesics on a manifold



Suppose we want the shortest path between two points A and B on a d-dimensional manifold.

Suppose infinitessimal distances on the manifold are given in terms of infinitessimal coordinate differences dx^i by

$$ds = \sqrt{\sum_{i,j=1}^{d} dx^i \, dx^j \, g_{ij}(\{x^k\})}$$

for some collection of functions $g_{ij}(\{x^k\})$ called the metric, which we can think of as a symmetric matrix, $g_{ij} = g_{ji}$.

Parametrizing paths by a parameter t, we can write for ds along the path:

$$ds = dt \sqrt{\sum_{i,j=1}^{d} \dot{x}^i(t) \, \dot{x}^j(t) \, g_{ij}(\{x^k\})} \equiv dt \sqrt{\dot{x}^2}$$

The length of the path from A to B is then,

$$S = \int_A^B ds = \int_{t_A}^{t_B} dt \sqrt{\sum_{i,j} \dot{x}^i \dot{x}^j g_{ij}}$$

The path of shortest distance stationarizes S.

$\delta S = 0 \rightarrow$ Euler-Lagrange Eqs.

$$\delta x^k: \quad \boxed{\frac{d}{dt}\left(\sum_j \frac{\dot{x}^j g_{kj}}{\sqrt{\dot{x}^2}}\right) - \frac{1}{2}\sum_{i,j} \frac{\dot{x}^i \dot{x}^j}{\sqrt{\dot{x}^2}}\left(\frac{\partial}{\partial x^k} g_{ij}\right) = 0}$$

With some manipulation and a definition we can simplify this equation for the geodesic.

$$\sum_j \frac{d}{dt}\left(\frac{\dot{x}^j}{\sqrt{\dot{x}^2}}\right) g_{kj} + \sum_j \frac{\dot{x}^j}{\sqrt{\dot{x}^2}} \frac{d}{dt}(g_{kj}) - \frac{1}{2}\sum_{i,j} \frac{\dot{x}^i \dot{x}^j}{\sqrt{\dot{x}^2}} \frac{\partial}{\partial x^k} g_{ij} = 0$$

$\Big\downarrow$ chain rule

$$\sum_j \frac{d}{dt}\left(\frac{\dot{x}^j}{\sqrt{\dot{x}^2}}\right) g_{kj} + \sum_j \frac{\dot{x}^j}{\sqrt{\dot{x}^2}} \sum_i \left(\frac{\partial}{\partial x^i} g_{kj}\right) \dot{x}^i - \frac{1}{2}\sum_{i,j} \frac{\dot{x}^i \dot{x}^j}{\sqrt{\dot{x}^2}} \frac{\partial}{\partial x^k} g_{ij} = 0$$

$\Big\downarrow$ using symmetry of $\dot{x}^i \dot{x}^j$ in $i \leftrightarrow j$.

$$\sum_j \frac{d}{dt}\left(\frac{\dot{x}^j}{\sqrt{\dot{x}^2}}\right) g_{kj} + \sum_{i,j} \frac{\dot{x}^i \dot{x}^j}{\sqrt{\dot{x}^2}} \frac{1}{2}\left(\frac{\partial}{\partial x^i} g_{kj} + \frac{\partial}{\partial x^j} g_{ki}\right) - \frac{1}{2}\sum_{i,j} \frac{\dot{x}^i \dot{x}^j}{\sqrt{\dot{x}^2}} \frac{\partial}{\partial x^k} g_{ij} = 0$$

Multiplying by the inverse "matrix" $(g^{-1})^{\ell k}$, such that

$$\sum_k (g^{-1})^{\ell k} g_{kj} = \delta^\ell_j \quad \text{Kronecker delta}$$
$$= \begin{cases} 1 & \text{if } \ell = j \\ 0 & \text{if } \ell \neq j \end{cases}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\dot{x}^\ell}{\sqrt{\dot{x}^2}}\right) + \sum_{i,j} \frac{\dot{x}^i \dot{x}^j}{\sqrt{\dot{x}^2}} \underbrace{\sum_k \frac{1}{2}(g^{-1})^{k\ell}\left(\frac{\partial}{\partial x^i} g_{kj} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij}\right)}_{\Gamma^\ell_{ij}} = 0$$

Defining $\Gamma^{\ell}_{ij} = \sum_k \frac{1}{2}(g^{-1})^{k\ell}\left(\frac{\partial}{\partial x^i} g_{kj} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij}\right)$,

the geodesic eqn. becomes,

$$\boxed{\frac{d}{dt}\left(\frac{\dot{x}^{\ell}}{\sqrt{\dot{x}^2}}\right) + \sum_{i,j} \frac{\dot{x}^i \dot{x}^j}{\sqrt{\dot{x}^2}} \Gamma^{\ell}_{ij} = 0}$$

If the parameter t is chosen so that $\sqrt{\dot{x}^2} = 1$ along the path, then t is called an <u>affine parameter</u> and we have the final simplification:

$$\boxed{\ddot{x}^{\ell} + \sum_{i,j} \dot{x}^i \dot{x}^j \Gamma^{\ell}_{ij} = 0} \qquad \text{Geodesic Eqn.}$$

Given the form of the metric g_{ij}, we could calculate Γ^{ℓ}_{ij} and, by solving the geodesic eqn, determine the paths of shortest distance.

For example, on a sphere in spherical coordinates,

$$ds^2 = R^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$\Rightarrow g_{ij}$ has components $\begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}$.

<u>Exercise</u>: The solutions to the geodesic equation in this case are the great circles, for example the equator $\theta = \pi/2$.