Noether's Theorem: Symmetries and Conservation Laws

Conservation laws are a class of first integrals of the equations of motion. They are represented by equations of the form $f(q_1,\dots,q_n;\dot{q}_1,\dots,\dot{q}_n,t)=\text{const.}$

The conserved quantity f depends on up-to first derivatives of the generalized coordinates, whereas the equations of motion are generally of second order in derivatives of the generalized coordinates.

Emmy Nöether discovered one of the most important results in classical mechanics: a relation between a large class of symmetries and conservation laws.

A symmetry is a transformation of the generalized coordinates $q_i(t)$ that leaves the action invariant.
Examples: Translations $x_i \to x_i + a_i$
Rotations $\vec{r} \to R\vec{r}$, where $R^T R = \mathbb{1}$

• Note: not every system respects these symmetries.

These are examples of continuous symmetries, parametrized by continuous parameters. They are smoothly connected to the identity (i.e. no transformation).

☆. Noether's Theorem: For every continuous symmetry connected to the identity, there is a conserved quantity.

Proof of Noether's Theorem

Consider a transformation of the form $q_i \to q_i + \epsilon \Delta q_i$, where Δq_i are functions of $\{q_k\}$ and t.

The factor of ϵ will remind us that we will be interested in expansions to first order in the transformation parameters.

To lowest order in ϵ, the action transforms as follows:

$$I = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$$

$$\longrightarrow I + \int_{t_1}^{t_2} dt \left[\sum_i \left(\frac{\partial L}{\partial q_i} \epsilon \Delta q_i + \frac{\partial L}{\partial \dot{q}_i} \epsilon \Delta \dot{q}_i \right) \right]$$

$= I$ if the transformation represents a symmetry.

In that case, $\displaystyle\int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} \Delta q_i + \frac{\partial L}{\partial \dot{q}_i} \Delta \dot{q}_i \right) dt = 0$

By Lagrange's equations, $\dfrac{\partial L}{\partial q_i} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i} \right)$. Hence,

$$\int_{t_1}^{t_2} \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \Delta q_i + \frac{\partial L}{\partial \dot{q}_i} \Delta \dot{q}_i \right) dt = 0.$$

$$= \int_{t_1}^{t_2} \frac{d}{dt} \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \Delta q_i \right) dt = 0$$

$$\Rightarrow \sum_i \frac{\partial L}{\partial \dot{q}_i} \Delta q_i \bigg|_{t_2} = \sum_i \frac{\partial L}{\partial \dot{q}_i} \Delta q_i \bigg|_{t_1} \quad \text{(for any } t_1, t_2\text{)}$$

Define $p_k \equiv \frac{\partial L}{\partial \dot{q}_k}$ canonical momentum conjugate to q_k.

We have shown that if $q_i \to q_i + \epsilon \Delta q_i$ leaves the action invariant, then the quantity

$$f = \boxed{\sum_i p_i \Delta q_i \quad \text{is conserved.}}$$

<u>Example</u>: Suppose $L(q_i, \dot{q}_i, t)$ is independent of one of the coordinates q_k.

Then $q_k \to q_k + \epsilon$ leaves L invariant, and is a symmetry of the system.

The corresponding conserved quantity is the canonical momentum p_k.

This also follows from Lagrange's equations:

If $L(q_j, \dot{q}_j, t)$ is independent of some q_k, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \cancel{\frac{\partial L}{\partial q_k}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

$$= \boxed{\frac{d}{dt} p_k = 0}$$

Example: Free particle $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

Translation invariance $x \to x + \epsilon$ leaves L invariant.

Conserved quantity $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

Similarly, translation invariance in y, z

$\rightarrow$ conserved quantities $p_y = m\dot{y}$

$p_z = m\dot{z}$

The mechanical momentum $\vec{p} = m\dot{\vec{r}}$ is conserved.

Example: Particle in an electromagnetic field.

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - q\,\phi(\vec{x}) + q\,\vec{A}\cdot\dot{\vec{r}}$$

electric charge — electric potential — vector potential

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}$$

Suppose $\phi(\vec{r})$ and $\vec{A}(\vec{r})$ are independent of the coordinate x.

The conserved momentum conjugate to x is

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x$$

Note that in the presence of magnetic fields, the conserved momentum is not the same as the kinetic (i.e. mechanical) momentum.

In such cases, Newton's 3rd law expressed as equality of action and reaction is violated.

Conservation of Energy

Conservation laws can also be derived in cases where the action is not invariant, but transforms only by the addition of boundary terms expressed as an integral of a total derivative with respect to time.

Consider a time translation: $t \to t+\epsilon$

$$q_i(t) \to q_i(t+\epsilon) = q_i(t) + \epsilon \dot{q}_i(t) + \mathcal{O}(\epsilon^2)$$
$$\dot{q}_i(t) \to \dot{q}_i(t+\epsilon) = \dot{q}_i(t) + \epsilon \frac{d}{dt}\dot{q}_i + \mathcal{O}(\epsilon^2)$$

The Lagrangian transforms as $L \to L + \epsilon \frac{dL}{dt} + \mathcal{O}(\epsilon^2)$

$$L \to L + \overbrace{\epsilon \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}\dot{q}_i \right) + \epsilon \frac{\partial L}{\partial t}}^{\epsilon \frac{dL}{dt}}$$

$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$ by the Lagrange Eqs.

$$L \to L + \epsilon \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}\dot{q}_i \right) + \epsilon \frac{\partial L}{\partial t}$$

$$= L + \epsilon \sum_i \frac{d}{dt}\left(\underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} \dot{q}_i \right) + \epsilon \frac{\partial L}{\partial t}$$

Equating the change in L with $\epsilon \frac{dL}{dt}$ gives:

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) = -\frac{\partial L}{\partial t}.$$

If the Lagrangian is not explicitly dependent on t, so that $\frac{\partial L}{\partial t} = 0$:

$$\boxed{\frac{d}{dt}\left(\sum_i p_i \dot{q}_i - L \right) = 0}$$

The conserved quantity is the Hamiltonian

$$H = \sum_i p_i \dot{q}_i - L$$

Example: $L = \frac{1}{2} m \dot{\vec{r}}^2 - V(\vec{r})$

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i$$

$$H = \sum_i m \dot{x}_i \dot{x}_i - \left(\frac{1}{2} m \sum_i \dot{x}_i \dot{x}_i - V(\vec{r}) \right)$$

$$= \frac{1}{2} m \dot{\vec{r}}^2 + V(\vec{r})$$

This is the conserved mechanical energy.