More About Lagrange's Eqs and Examples

– Goldstein 1.4 +

- Note that the choice of Lagrangian that leads to a particular set of equations of motion is not unique:

Define $L'(q,\dot{q},t) \equiv L(q,\dot{q},t) + \dfrac{dF(q,t)}{dt}$

$$= L(q,\dot{q},t) + \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t}$$

$$\frac{\partial}{\partial \dot{q}}\left(\frac{dF}{dt}\right) = \frac{\partial F}{\partial q}\ , \quad \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left(\frac{dF}{dt}\right)\right) = \frac{\partial^2 F}{\partial q^2}\dot{q} + \frac{\partial^2 F}{\partial q\,\partial t}$$

Also, $$\frac{\partial}{\partial q}\left(\frac{dF}{dt}\right) = \frac{\partial^2 F}{\partial q^2}\dot{q} + \frac{\partial^2 F}{\partial q\,\partial t}$$

Hence, $$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left(\frac{dF}{dt}\right)\right) - \frac{\partial}{\partial q}\left(\frac{dF}{dt}\right) = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) - \frac{\partial L'}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}}$$, as claimed.

- If nonconservative forces are present, we can always write Lagrange's eqs. as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \quad , \text{ where}$$

$L = T - V$ contains the potential for conservative forces and Q_j describes the nonconservative forces.

- For velocity-dependent nonconservative forces, it may be possible to write $Q_j = -\frac{\partial F}{\partial \dot{q}_j}$ for some $F(\{\dot{q}_j\})$.

Then, Lagrange's eqs. become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial F}{\partial \dot{q}_j} = 0$$

Examples of the Lagrangian formalism:

Example:

Free particle in Cartesian coordinates

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad , \quad V = 0.$$

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0.$$

Eqs of motion:

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= m\ddot{x} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= m\ddot{y} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= m\ddot{z} = 0. \end{aligned}\right\} m\ddot{\vec{r}} = 0$$

Example: Particle w/ Conservative force $\vec{F} = -\nabla V(\vec{r})$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x,y,z)$$

Eqs. of motion:

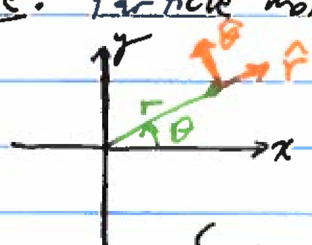
$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= m\ddot{x} + \frac{\partial V}{\partial x} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= m\ddot{y} + \frac{\partial V}{\partial y} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= m\ddot{z} + \frac{\partial V}{\partial z} = 0 \end{aligned}\right\} \boxed{m\ddot{\vec{r}} = -\nabla V}$$

Generalized forces: $Q_x = \vec{F}\cdot\frac{\partial \vec{r}}{\partial x} = F_x$

$$Q_y = \vec{F}\cdot\frac{\partial \vec{r}}{\partial y} = F_y$$

$$Q_z = \vec{F}\cdot\frac{\partial \vec{r}}{\partial z} = F_z$$

Example: Particle motion in plane polar coordinates



$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$
$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases}$$

Kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$$

Generalized forces in polar coordinates:

$$\vec{r} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y} + z\,\hat{z}$$

$$\rightarrow \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} = \hat{r}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\hat{x} + r\cos\theta\,\hat{y} = r\,\hat{\theta}$$

$$\frac{\partial \vec{r}}{\partial z} = \hat{z}$$

$$\vec{F} = F_x\,\hat{x} + F_y\,\hat{y} + F_z\,\hat{z} = F_r\,\hat{r} + F_\theta\,\hat{\theta} + F_z\,\hat{z}$$

Generalized force $Q_r = \vec{F}\cdot\frac{\partial \vec{r}}{\partial r} = \vec{F}\cdot\hat{r} = F_r$

$$Q_\theta = \vec{F}\cdot\frac{\partial \vec{r}}{\partial \theta} = \vec{F}\cdot(r\,\hat{\theta}) = r\,F_\theta$$

$$Q_z = \vec{F}\cdot\frac{\partial \vec{r}}{\partial z} = F_z$$

Lagrange Eqs: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r$

$$\rightarrow \boxed{m\ddot{r} - m\underbrace{r\dot{\theta}^2}_{\text{centripetal acceleration}} = F_r}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

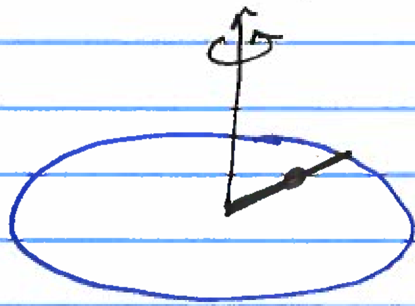
$$\rightarrow \boxed{\frac{d}{dt}\left(\underbrace{mr^2\dot{\theta}}_{\text{angular momentum}}\right) = \underbrace{r\,F_\theta}_{\text{torque}}}$$

$$\frac{d}{dt}\left(\text{angular momentum}\right) = \text{torque}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = Q_z$$

$$\rightarrow \boxed{m\ddot{z} = F_z}$$

Example: Bead on a straight rotating wire



$\dot{\theta} = \omega$ fixed

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \omega^2)$$, No generalized forces.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = \boxed{m\ddot{r} - m r \omega^2 = 0}$$

Solution: Ansatz $r(t) = e^{\alpha t}$

$\rightarrow m\alpha^2 r - m\omega^2 r = 0 \rightarrow \alpha^2 = \omega^2$

$\alpha = \pm \omega$

Linear homogeneous diff eq. $\rightarrow$ sol'n $r(t) = Ae^{\omega t} + Be^{-\omega t}$

Initial conditions: Suppose $r(0) = r_0 \rightarrow A + B = r_0$

$\dot{r}(0) = 0 \rightarrow A\omega \mp B\omega = 0$

$\rightarrow A = B$

$$\Rightarrow r(t) = \frac{r_0}{2}(e^{\omega t} + e^{-\omega t})$$

$$\boxed{r(t) = r_0 \cosh \omega t} \xrightarrow{\omega t \to \infty} \frac{r_0}{2} e^{\omega t}$$

Why the motion away from the center?

The constraint force applied by the wire is in the angular direction. There is no force in the radial direction, so the centripetal acceleration must be compensated by an equivalent radial acceleration in the opposite direction.

To calculate the force of constraint of the wire on the bead, we can use $\vec{N} = \frac{d\vec{L}}{dt}$.

Angular momentum $L = mr^2\omega = m\omega r_0^2 (\cosh(\omega t))^2$

$$\frac{dL}{dt} = 2m\omega r_0^2 \cosh \omega t \sinh \omega t$$

$$= (r_0 \cosh(\omega t)) f$$

constraint force

$$\rightarrow \boxed{f = 2m\omega r_0 \sinh(\omega t)}$$

$$\xrightarrow{\omega t \to \infty} m\omega r_0 e^{\omega t} \approx 2m\omega r(t)$$

The constraint force required to hold the bead on the wire increases with the bead's displacement from the center of the circular motion.

This is because the centripetal acceleration increases with radial distance.