Relativistic Kinematics

Four-Vector Momentum: Generalization of $\vec{p} = m\frac{d\vec{x}}{dt}$

Let x^μ be the spacetime coordinates in some inertial frame.
parametrize the particle trajectory by $\vec{x}(t)$ or $\vec{x}(\tau)$

τ = proper time elapsed along particle trajectory.

Recall $dt = \gamma\, d\tau$ ← "Moving clocks run slow"

Define $p^\mu = m\frac{dx^\mu}{d\tau}$ ← transforms like a 4-vector. $d\tau$ is Lorentz invariant.

$$p^0 = m\frac{dx^0}{d\tau} = mc\gamma$$

$$p^i = m\frac{dx^i}{d\tau} = m\vec{v}\gamma \quad v^i = \frac{dx^i}{dt}$$

In the nonrelativistic limit, $p^0 \approx mc\left(1 + \frac{1}{2}\frac{\vec{v}^2}{c^2}\right) = mc + \frac{\frac{1}{2}m\vec{v}^2}{c}$

mc is a constant independent of $\vec{v}$.
$\frac{1}{2}m\vec{v}^2$ is the nonrelativistic kinetic energy.
⇒ Identify $p^0 = E/c$, where E = relativistic energy

We have $\boxed{E = mc^2\gamma} = \underset{\text{rest energy}}{mc^2} + \underbrace{mc^2(\gamma - 1)}_{\equiv T \text{ kinetic energy.}}$

Spatial components of 4-momentum: $\vec{p} = m\frac{d\vec{x}}{d\tau} = m\frac{d\vec{x}}{dt}\gamma$

nonrelativistic limit: $\vec{p} \approx m\frac{d\vec{x}}{dt} = m\vec{v}$. = nonrelativistic momentum.

Just as $ds^2 = dx^\mu dx^\nu \eta_{\mu\nu}$ is Lorentz invariant,

so is
$$p^\mu p^\nu \eta_{\mu\nu} = (p^0)^2 - \vec{p}^2$$
$$= (mc\gamma)^2 - (m\vec{v}\gamma)^2$$
$$= \frac{m^2c^2}{1-v^2/c^2}\left(1-v^2/c^2\right)$$
$$= m^2c^2$$

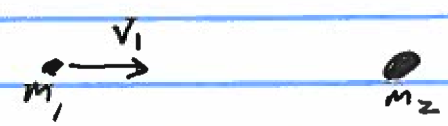
In other words, $E^2/c^2 = \vec{p}^2 + m^2c^2$,

or $\boxed{E^2 = \vec{p}^2c^2 + m^2c^4}$ ← Relativistic relation between Energy and momentum.

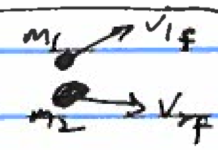
Spacetime translation invariance → Conservation of (total) E and $\vec{p}$.

<u>Kinematics of Collisions</u>

Lab frame: One particle is at rest initially.



Before collision



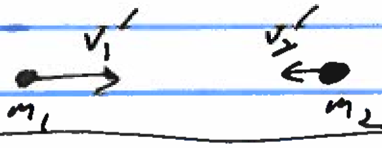
After collision

Initial total 4-momentum:

suppose velocity of m_1 is in x-direction.

$$p^\mu = (m_1 c\gamma_1 + m_2 c,\ m_1\gamma_1 v_1,\ 0,\ 0)$$

In the Center-of-Momentum frame, the total $\vec{p}\,' = \vec{0}$.
(But the energy $p'^0/c \neq 0$.)



$$\boxed{\vec{p}\,' = m_1 \vec{v}_1\,'\gamma_1' + m_2 \vec{v}_2\,'\gamma_2' = \vec{0}}$$

$$\boxed{p'^\mu = (m_1\gamma_1' c + m_2\gamma_2' c,\ 0,\ 0,\ 0)}$$

The boost $\vec{\beta}\,'$ to go from the lab to the COM frame:

$$\vec{\beta}\,' = -\vec{\beta}_2\,'$$ (because $\vec{\beta}_2 = 0$ — initial velocity of m_2 in lab frame)

Velocity $\vec{\beta}_1\,' c$ of mass m_1 in COM frame in terms of $\vec{\beta}_1$ in lab frame and boost $\vec{\beta}\,'$ — velocity addition.

$$\boxed{\beta_1' = \frac{\beta_1 - \beta'}{1 - \beta_1\beta'}}$$

To find the boost velocity β' relating the two frames, we can use the Lorentz invariance of $p^\mu p^\nu \eta_{\mu\nu}$:

In the lab frame,

$$(p^0)^2 - \vec{p}^{\,2} = (m_1 c\gamma_1 + m_2 c)^2 - (m_1 v_1 \gamma_1)^2 = (m_1^2 + m_2^2 + 2m_1 m_2 \gamma_1)c^2$$
$$= (m_1^2 + m_2^2)c^2 + 2m_2 E_1$$

$E_1 = m_1 c^2 \gamma_1$

In the COM frame:

$$(p'^0)^2 - (\vec{p}\,')^2 = (m_1 \gamma_1' c + m_2 \gamma_2' c)^2$$

$$= \left\{ -\frac{m_2 \gamma_2' \beta_2' c}{\beta_1'} + m_2 \gamma_2' c \right\}^2 \quad \text{using } m_1 \vec{\beta}_1' \gamma_1' = -m_2 \vec{\beta}_2' \gamma_2'$$

$$= m_2^2 (\gamma')^2 c^2 \left\{ 1 + \frac{\beta' (1 - \beta_1 \beta')}{\beta_1 - \beta'} \right\}^2 \quad \text{using } \beta_1' = \frac{\beta_1 - \beta'}{1 - \beta_1 \beta'} \text{ and } \beta_2' = -\beta'$$

$$= m_2^2 (\gamma')^2 c^2 \left\{ \frac{\beta_1 (1 - (\beta')^2)}{\beta_1 - \beta'} \right\}^2$$

$$= \boxed{\frac{m_2^2 c^2 \beta_1^2 (1 - (\beta')^2)}{(\beta_1 - \beta')^2}} \quad \text{using } (\gamma')^2 = \frac{1}{1 - (\beta')^2}$$

This must be the same as $(p^0)^2 - \vec{p}^{\,2}$ in the lab frame.

Hence, the boost from the lab to COM frame is determined by

$$\boxed{\frac{m_2^2 c^2 \beta_1^2 (1 - (\beta')^2)}{(\beta_1 - \beta')^2} = \left[m_1^2 + m_2^2 + \frac{2 m_1 m_2}{\sqrt{1 - \beta_1^2}} \right] c^2}$$

Particle Production: Consider the nuclear reaction $p+n \to p+n+p+\bar{p}$

p = proton, $\bar{p}$ = antiproton, n = neutron.

$m_p = 938$ MeV, similar to m_n.

— treat all masses as equal.

Question: If the initial neutron is at rest in the lab frame what is the minimum required kinetic energy of the proton in order to create the $p\bar{p}$ pair?

Total 4-momentum.

Initially: In lab frame calculate $p^\mu p^\nu \eta_{\mu\nu}$:

$$(p^0)^2 - (\vec{p})^2 = (m_p^2 + m_n^2)c^2 + 2 m_n \underbrace{m_p c^2 \gamma_p}_{E_p}$$

E_p — proton energy.

Write $E_p = m_p c^2 + T_p$

proton kinetic energy

$$\Rightarrow \boxed{(p^0)^2 - (\vec{p})^2 = (m_p + m_n)^2 c^2 + 2 m_n T_p}$$

At threshold for the reaction (i.e. minimum T_p), in the COM frame all final particles are at rest.

$$\Rightarrow \boxed{(p^{0\prime})^2 - (\vec{p}\,')^2 = (3m_p + m_n)^2 c^2} = (p^0)^2 - (\vec{p})^2$$

by conservation of p^μ and Lorentz invariance of $(p^0)^2 - (\vec{p})^2$.

$$\to T_p = \frac{[(3m_p+m_n)^2 - (m_p+m_n)^2]c^2}{2m_n} = \frac{[8m_p^2 + 4m_p m_n]c^2}{2m_n}$$

$$\boxed{T_p \approx 6 m_p c^2 = 5.63 \text{ GeV}}$$

This is 3× the difference in final and initial rest energies!