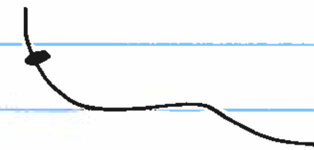
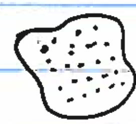
Goldstein 1.3

Systems with Constraints

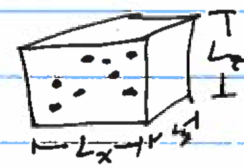
Examples of constraints:

- Bead on a wire



- Rigid body  $|\vec{r}_i - \vec{r}_j|^2 = $ fixed.

- Gas inside a container



Classification of constraints:

Constraints are **holonomic** if they can be written in the form $f(\vec{r}_1, \vec{r}_2, \ldots, t) = 0$.

Examples: • Bead on a wire: equations of the wire's curve are the constraint.

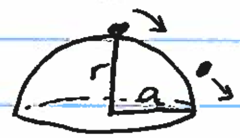
• Rigid body $|\vec{r}_i - \vec{r}_j|^2 - c_{ij}^2 = 0$

↑ constants.

Nonholonomic constraints cannot be written in the form $f(\vec{r}_i, t) = 0$

Examples: • Gas inside a container $\begin{cases} 0 < x_i < L_x \\ 0 < y_i < L_y \\ 0 < z_i < L_z \end{cases}$ Inequalities - Nonholonomic

• Bead placed on a sphere:



$r^2 - a^2 \geq 0$ Inequality

• Velocity-dependent constraints.

Rheonomous constraints — depend explicitly on time

Scleronomous constraints — do not depend explicitly on time.

Difficulties posed by constraints:

1) Coordinates $\vec{r}_i$ are not all independent
2) Forces of constraint, e.g. force of wire on a bead, is not known a priori.

For holonomic constraints, the coordinates can be replaced by a smaller number of independent **generalized coordinates**

N particles w/o constraints: 3N independent coordinates (degrees of freedom)

K holonomic constraint eqs: $-K$ degrees of freedom

$3N-K$ generalized coordinates q_i required to describe the motion.

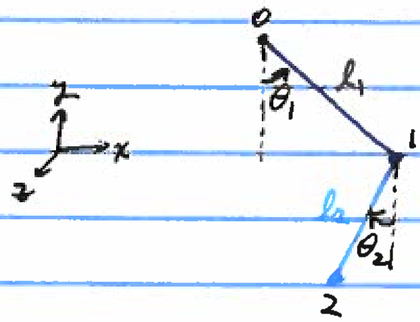
$$\vec{r}_1 = \vec{r}_1(q_1, q_2, \ldots, q_{3N-K}, t)$$

$$\vdots$$

$$\underset{\substack{\uparrow \\ \text{Cartesian} \\ \text{coordinates}}}{\vec{r}_N} = \vec{r}_N(\underbrace{q_1, q_2, \ldots, q_{3N-K}}_{\substack{\text{Generalized} \\ \text{coordinates.}}}, t)$$

The art of solving for the motion of systems with constraints lies in identifying suitable generalized coordinates.

Example: Double Pendulum



2 particles: $3 \times 2 = 6$ Cartesian coordinates

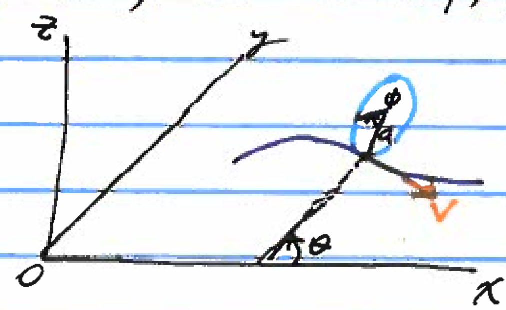
$$\left.\begin{array}{l} \text{Motion in plane: } z_1 = z_2 = 0 \\ \text{Pendulum: } |\vec{r}_1|^2 = l_1^2 \\ \qquad |\vec{r}_2 - \vec{r}_1|^2 = l_2^2 \end{array}\right\} \text{4 constraints}$$

$\Rightarrow 6 - 4 = 2$ generalized coordinates

Convenient choice is θ_1, θ_2 as in the figure.

Nonholonomic Constraints: Constraint equations do not eliminate the dependent coordinates.

Example: Rolling without slipping.



Vertical disk rolling on horizontal x-y plane. $\vec{V}$ = velocity of center of disk.

Motion described by coordinates x, y — center of disk

ϕ — rotation about axis of disk

θ — $\measuredangle$ between axis of disk and x-axis.

Rolling constraint: $v = a\dot{\phi}$

$a \equiv$ radius of disk.

$$\left.\begin{array}{l} \dot{x} = v \sin\theta \\ \dot{y} = -v \cos\theta \end{array}\right\} \text{Direction perpendicular to disk axis.}$$

→ Differential equations of constraint:

$$\begin{cases} dx - a\sin\theta\, d\phi = 0 \\ dy + a\cos\theta\, d\phi = 0 \end{cases}$$

These constraints are not integrable:
We might seek a function $f(x, y, \theta, \phi)$ such that

$$f\,dx - f\,a\sin\theta\, d\phi = dg(x, y, \theta, \phi) \quad (= 0 \text{ constraint})$$

for some function g. Then the first constraint equation could be integrated.

For an exact differential $dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial \theta}d\theta + \frac{\partial g}{\partial \phi}d\phi$, equality of mixed partial derivatives restricts the form of the coefficients of dx, dy, $d\theta$ and $d\phi$:

Suppose $f\,dx - a\sin\theta\, f\, d\phi = dg$ for some g.
Then $\frac{\partial g}{\partial x} = f$, $\frac{\partial g}{\partial y} = 0$, $\frac{\partial g}{\partial \theta} = 0$, $\frac{\partial g}{\partial \phi} = -a\sin\theta\, f$

$$\left.\begin{aligned} \frac{\partial^2 g}{\partial x\,\partial\theta} &= \frac{\partial^2 g}{\partial\theta\,\partial x} \rightarrow \boxed{\frac{\partial f}{\partial \theta} = 0} \\ \frac{\partial^2 g}{\partial\phi\,\partial\theta} &= \frac{\partial^2 g}{\partial\theta\,\partial\phi} \Rightarrow \boxed{\frac{\partial}{\partial\theta}(-a\sin\theta\, f) = 0} \end{aligned}\right\} \text{ not compatible.}$$

Hence, no integrating function f exists, and the constraint is not integrable, as claimed.
(Note that for $\theta = \pi/2$ → the dx constraint can be integrated; similarly, for $\theta = 0$, the dy constraint can be integrated.)

Sec. 1.4 D'Alembert's Principle and Lagrange's Equations

Consider an arbitrary infinitessimal displacement $\delta\vec{r}_i$ consistent with the forces and constraints. This need not describe the actual motion of the system, so the displacement $\delta\vec{r}_i$ is called virtual.

Equilibrium: $\vec{F}_i = \vec{0}$, $\sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$

Write $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$

applied force. force of constraint

$$\sum \vec{F}_i^{(a)} \cdot \delta\vec{r}_i + \sum \vec{f}_i \cdot \delta\vec{r}_i = 0$$

Suppose the net virtual work of the forces of constraint $= 0$, e.g. bead on wire: constraint force is perpendicular to bead motion.
(not true if there is sliding friction.)

$\Rightarrow$ Equilibrium $\rightarrow$ $\sum \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0$ Principle of virtual work.

What if the system is not in equilibrium?

$$\vec{F}_i = \dot{\vec{p}}_i \Rightarrow \vec{F}_i - \dot{\vec{p}}_i = \vec{0}.$$

$$\rightarrow \sum(\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$$

$$\sum(\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i + \underbrace{\sum \vec{f}_i \cdot \delta\vec{r}_i = 0}$$

again assume forces of constraint do no work.

$$\rightarrow \boxed{\sum(\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0}$$ D'Alembert's Principle.

We would like to express D'Alembert's principle in terms of the independent generalized coordinates.

Suppose we have $\vec{r}_i = \vec{r}_i(q_1, q_2, \dots, q_n, t)$

$$\vec{v}_i \equiv \frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \qquad \text{chain rule.}$$

Also, $\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$.

Virtual work $\sum_i \vec{F}_i \cdot \delta \vec{r}_i = \sum_{i,j} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$

$$= \sum_j Q_j \delta q_j$$

$$Q_j \equiv \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \qquad \text{Generalized force.}$$

Note: Generalized coords. do not need to have dims of length (e.g. angles), and Q_j does not need to have dimensions of force.

Now consider the term $\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i$ in D'Alembert's principle

$$\sum \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i$$

$$= \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$= \sum_j \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right] \delta q_j$$

In the last term, use

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \, \partial t}$$

$$= \frac{\partial \vec{v}_i}{\partial q_j} \qquad \left(\text{Recall } \vec{v}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)$$

Also, $\dfrac{\partial \vec{v}_i}{\partial \dot{q}_j} = \dfrac{\partial \vec{r}_i}{\partial q_j}$

$$\Rightarrow \sum m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\underbrace{\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right)}_{\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m_i v_i^2\right)} - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j}\right]$$

Back to D'Alembert's Principle:

$$\sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j - \sum_j Q_j \delta q_j = 0$$

$$\sum_j \left\{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\underbrace{\left(\sum_i \frac{1}{2}m_i v_i^2\right)}_{\equiv T}\right] - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2}m_i v_i^2\right) - Q_j\right\}\delta q_j = 0$$

$$\boxed{\sum_j \left\{\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] - Q_j\right\}\delta q_j = 0}$$

For holonomic constraints, q_j are chosen to be an <u>independent</u> set of generalized coords $\rightarrow \delta q_j$ independent.

$$\rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j}$$

Now suppose $\vec{F}_i = -\nabla_i V$ for some scalar potential function $V(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, t)$.

Then $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$

D'Alembert

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$

We assume V is a fn. of q_j, but not $\dot{q}_j$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$

Define the <u>Lagrangian</u> $L = T - V$

D'Alembert's principle $\rightarrow$ $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$

<u>Lagrange's Eqs.</u>