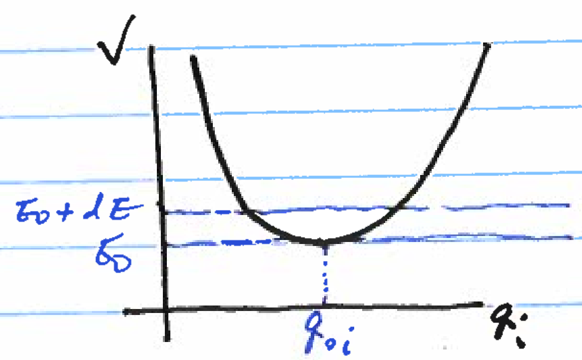
Oscillations - Goldstein Ch. 6

Systems disturbed from stable equilibrium generally evolve by way of small oscillations. In 1D we can describe these oscillations in terms of the Simple Harmonic Oscillator.
Here we consider systems with more than one degree of freedom.

Consider a conservative system, with potential depending on generalized coordinates q_i, $V = V(\{q_i\})$.

Equilibrium: $Q_i \equiv -\left.\frac{\partial V}{\partial q_i}\right|_{q_{0i}} = 0$

generalized forces. coordinates @ equilibrium



Stable equilibrium: q_i remains bounded after displacement from q_{0i}

Write $q_i = q_{0i} + \eta_i$

$$V(q_1, \ldots, q_n) = V(q_{01}, \ldots, q_{0n}) + \cancel{\left.\frac{\partial V}{\partial q_i}\right|_{q_{0i}} \eta_i} + \frac{1}{2} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{q_{0i}} \eta_i \eta_j + \ldots$$

0

To lowest nonvanishing order in η_i:

$$\boxed{V = \frac{1}{2} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{q_{0i}} \eta_i \eta_j \equiv \frac{1}{2} V_{ij} \eta_i \eta_j}$$

where

$$V_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{q_{0i}}$$

Note $V_{ij} = V_{ji}$ — like a symmetric matrix.

Kinetic energy:

Assume $T = \frac{1}{2} m_{ij}(\{q_i\}) \dot{q}_i \dot{q}_j$

$$= \frac{1}{2} m_{ij}(\{q_i\}) \dot{\eta}_i \dot{\eta}_j$$

Expand: $m_{ij}(q_1, \dots, q_n) = \underbrace{m_{ij}(q_{01}, \dots, q_{0n})}_{\equiv T_{ij}} + \left.\frac{\partial m_{ij}}{\partial q_k}\right|_{q_{0i}} \eta_k + \dots$

To lowest order, we can replace m_{ij} by

$$T_{ij} \equiv m_{ij}(q_{01}, \dots, q_{0n})$$

$$\boxed{T = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j}$$

$T_{ij} = T_{ji}$ independent of $\{\eta_i\}$.

$$\boxed{L = T - V = \frac{1}{2}\left(T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j\right)}$$

Euler-Lagrange Eqs for η_i:

$$\boxed{T_{ij}\,\ddot{\eta}_j + V_{ij}\,\eta_j = 0}$$

(If coordinates are such that T_{ij} is diagonal, we can write

$$T_i\,\ddot{\eta}_i + V_{ij}\,\eta_j = 0. \quad)$$

<u>Solution</u>: Assume an ansatz

$$\eta_i = \underbrace{C}_{\text{constant}}\, a_i\, e^{-i\omega t}$$

(motion described by real part of η_i)

$$\boxed{-\omega^2 T_{ij}\, a_j + V_{ij}\, a_j = 0},$$

or $(-\omega^2 T_{ij} + V_{ij})\, a_j = 0$

— n linear homogeneous eqs. for n a_j. Solution other than $a_j = 0\ \forall j$ requires

$$\det \begin{pmatrix} -\omega^2 T_{11} + V_{11} & -\omega^2 T_{12} + V_{12} & \cdots \\ -\omega^2 T_{21} + V_{21} & -\omega^2 T_{22} + V_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = 0.$$

$\rightarrow$ n^{th} order algebraic eqn for ω^2.
$\rightarrow$ n roots.

Suppose $\underline{V}\,\vec{a}_m = \omega_m^2\,\underline{T}\,\vec{a}_m$ (no sum over m.)

$\uparrow$ m^{th} "eigenvector"

$\omega_m^2 =$ m^{th} "eigenvalue"

The solutions for ω_m^2 and $\vec{a}_m$ are real.
($\vec{a}_m$ can be multiplied by an overall phase.)

Conjugate the eigenvalue-like equation:

$$\vec{a}_n^{\,+}\,\underline{V} = \omega_n^{2*}\,\vec{a}_n^{\,+}\,\underline{T} \qquad (\text{recall } \underline{V}^+ = \underline{V},\ \underline{T}^+ = \underline{T})$$

Multiply by $\vec{a}_m$ on the right, and the original equation by $\vec{a}_n^{\,+}$ on the left:

$$\boxed{(\omega_m^2 - \omega_n^{2*})\,\vec{a}_n^{\,+}\,\underline{T}\,\vec{a}_m = 0}$$

If $m=n$, $\rightarrow (\omega_m^2 - \omega_m^{2*})\underbrace{\vec{a}_m^{\,+}\,\underline{T}\,\vec{a}_m}_{=(\vec{a}_m^{\,+}\,T\,\vec{a}_m)^*} = 0$

writing $\vec{a}_m = \vec{\alpha}_m + i\vec{\beta}_m$, $\vec{\alpha}, \vec{\beta}$ real,

$$\vec{a}_m^{\,+}\,\underline{T}\,\vec{a}_m = \vec{\alpha}_m^{T}\,\underline{T}\,\vec{\alpha}_m + \vec{\beta}_m^{T}\,\underline{T}\,\vec{\beta}_m$$

2x kinetic energy if $\dot{\vec{q}}_m = \vec{\alpha}_m$ $\uparrow$ $\uparrow$ 2x kinetic energy if $\dot{\vec{q}}_m = \vec{\beta}_m$.

kinetic energy >0 for nonvanishing velocity

$\Rightarrow \boxed{\vec{a}_m^{\,+}\,\underline{T}\,\vec{a}_m > 0}$

$\rightarrow \boxed{\omega_m^2 = \omega_m^{2*}} \rightarrow$ Ratios of eigenvector components are real.

$\rightarrow$ $\boxed{\text{Can choose } \vec{a}_m \text{ real.}}$ (phase can be absorbed in C.)

Given $\vec{a}_m$, we have

$$\boxed{\omega_m^2 = \frac{\vec{a}_m^T \underline{V} \vec{a}_m}{\vec{a}_m^T \underline{T} \vec{a}_m}}$$

Stability requires $\vec{a}_m^T \underline{V} \vec{a}_m > 0$ — local minimum of potential.

Consider $(\omega_m^2 - \omega_n^2) \vec{a}_m^T \underline{T} \vec{a}_n = 0$.

If $\omega_m^2 \neq \omega_n^2$, $\boxed{\vec{a}_m^T \underline{T} \vec{a}_n = 0 \quad \text{for } m \neq n}$

We can choose $\boxed{\vec{a}_m^T \underline{T} \vec{a}_m = 1}$ — normalization.

Define the matrix of eigenvectors $\underline{A}_{im} = (a_m)_i$

$$\underline{A}^T \underline{T} \underline{A} = \mathbb{1}$$

i.e. $\underline{A}$ transforms $\underline{T}$ to $\mathbb{1}$ by a <u>congruence transformation</u>.

Define a diagonal matrix $\underline{w}^2$ with elements
$w^2_{im} = w_m^2 \delta_{im}$.

$$V_{ij} A_{jm} = T_{ij} A_{jl} w^2_{lm} \text{ , or}$$

$$\underline{V}\,\underline{A} = \underline{T}\,\underline{A}\,\underline{w}^2$$

Multiply on the left by $\underline{A}^T$:

$$\underline{A}^T \underline{V}\,\underline{A} = \underbrace{\underline{A}^T \underline{T}\,\underline{A}}_{=\underline{1}}\,\underline{w}^2$$

$$\rightarrow \boxed{\underline{A}^T \underline{V}\,\underline{A} = \underline{w}^2}$$

— $\underline{A}$ transforms $\underline{V}$ by a congruence into a diagonal matrix w/ the eigenvalues w_m^2 on the diagonals.

Summary: We can choose normalized Cartesian coordinates such that $\underline{T}_{ij} = \delta_{ij}$, such that the solutions for $\vec{q}$ and w^2 follow from the solution to

$$\underline{A}^T \underline{A} = \underline{1} \text{ , } \underline{A}^T \underline{V}\,\underline{A} = \underline{V}_{diag}$$

Solution: $\det(\underline{V} - w^2 \mathbb{1}) = 0$

Example: $L = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} V_{ij} x_i x_j$

constants

$$\det \begin{pmatrix} V_{11} - \lambda & V_{12} \\ V_{21} & V_{22} - \lambda \end{pmatrix} = 0, \qquad \lambda = \omega^2$$

Solutions: $\lambda = \frac{1}{2}\left(V_{11} + V_{22} \pm \sqrt{(V_{11} - V_{22})^2 + 4 V_{12} V_{21}}\right)$

Eigenvectors are found by substituting the eigenvalues into

$$\underline{V} \vec{a}_n = \lambda_n \vec{a}_n$$

Suppose $V_{11} > V_{22} > 0$, $0 \neq V_{12} = V_{21} \ll V_{11} - V_{22}$.

Define

$$\delta \equiv \frac{V_{12}}{V_{11} - V_{22}} \ll 0 \quad \text{(by assumption)}$$

Eigenvalues: $\lambda_1 \approx V_{11} + V_{12}\delta$

$\lambda_2 \approx V_{22} - V_{12}\delta$

Eigenvectors: $\begin{pmatrix} -V_{12}\delta & V_{12} \\ V_{21} & V_{22} - V_{11} - V_{12}\delta \end{pmatrix} \begin{pmatrix} (a_1)_1 \\ (a_1)_2 \end{pmatrix} = 0$

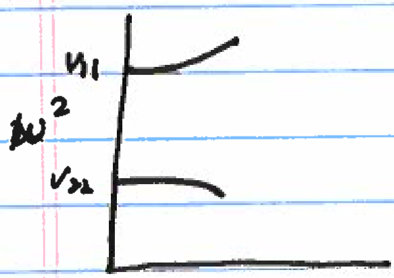
$$\vec{a}_1 = \begin{pmatrix} 1 - \frac{\delta^2}{2} \\ \delta - \frac{\delta^3}{2} \end{pmatrix} + \text{higher order in } \delta.$$

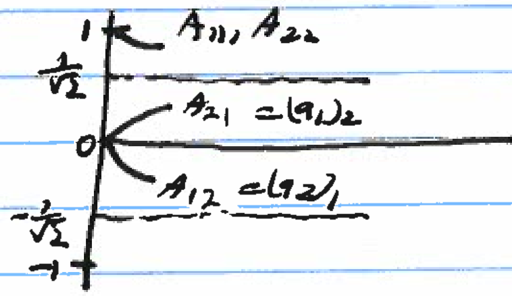
Similarly,

$$\vec{a}_2 = \begin{pmatrix} -\delta + \frac{\delta^3}{2} \\ 1 - \frac{\delta^2}{2} \end{pmatrix}$$

$$\underline{A} = \begin{pmatrix} (a_1)_1 & (a_2)_1 \\ (a_1)_2 & (a_2)_2 \end{pmatrix} \approx \begin{pmatrix} 1 - \frac{\delta^2}{2} & -\delta + \frac{\delta^3}{2} \\ \delta - \frac{\delta^3}{2} & 1 - \frac{\delta^2}{2} \end{pmatrix}$$

$$\underline{A}_{11} \approx \underline{A}_{22}, \quad \underline{A}_{12} \approx -\underline{A}_{21}$$





$$\delta \approx \frac{\nu_{12}}{\nu_{11} - \nu_{12}}$$

Note! In cases where $\exists$ degenerate eigenvalues, we can still choose all eigenvectors to be orthogonal.

Frequencies of Free Vibration, Normal Coordinates

Summary: Oscillations of system of n generalized coords. about equilibrium are characterized by n frequencies of free vibration.

$\equiv$ resonant frequencies.

General solution: $$\boxed{\eta_i = \sum_{k=1}^{n} C_k (a_k)_i \, e^{-i\omega_k t} + \text{complex conjugate.}}$$

– Can be written $$\eta_i(t) = \sum_{k=1}^{n} \underbrace{f_k}_{\text{amplitude}} (a_k)_i \cos(\omega_k t + \underbrace{\delta_k}_{\text{phase}})$$

The amplitudes and phases are determined by initial conditions.

$$t=0: \quad \eta_i(0) = 2\,\mathrm{Re}\left(C_k (a_k)_i\right)$$
$$\dot{\eta}_i(0) = 2\,\mathrm{Im}\left(C_k (a_k)_i \,\omega_k\right)$$
$\Big\}$ $2n$ eqs. for n complex C_k.

Note that for generic free vibration frequencies, $\eta_i(t)$ is not periodic.

However, we can define new periodic generalized coords. — Normal coordinates.

$\eta_i = A_{ij}\zeta_j$, or $\vec{\eta} = \underline{A}\vec{\zeta}$

$$V = \frac{1}{2}\vec{\eta}^T \underline{V}\vec{\eta}$$
$$= \frac{1}{2}(\underline{A}\vec{\zeta})^T \underline{V}(\underline{A}\vec{\zeta})$$

$$= \frac{1}{2}\vec{\zeta}^T \underbrace{\underline{A}^T\underline{V}\,\underline{A}}_{=\underline{\omega}^2}\vec{\zeta}$$

$= \underline{\omega}^2$ matrix of eigenvalues = resonance frequencies.

$$\boxed{V = \frac{1}{2}\sum_k \omega_k^2 \zeta_k^2}$$

Kinetic energy: $T = \frac{1}{2}\dot{\vec{\eta}}^T \underline{T}\dot{\vec{\eta}}$

$$= \frac{1}{2}\dot{\vec{\zeta}}^T \underbrace{\underline{A}^T\underline{T}\,\underline{A}}_{=\mathbb{1}}\dot{\vec{\zeta}}$$

$$= \frac{1}{2}\dot{\vec{\zeta}}^T\dot{\vec{\zeta}}$$

$$\boxed{T = \frac{1}{2}\sum_i \dot{\zeta}_i\dot{\zeta}_i}$$

The matrix $\underline{A}$ simultaneously diagonalizes $\underline{T}$ and $\underline{V}$.

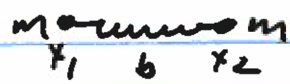
$$L = \frac{1}{2}\sum_k(\dot{\zeta}_k\dot{\zeta}_k - \omega_k^2\zeta_k^2)$$

Euler-Lagrange eqs: $\ddot{\zeta}_k + \omega_k^2\zeta_k = 0$

solutions: $\zeta_k = C_k e^{-i\omega_k t}$ periodic

— Normal modes of vibration.

Example: Linear Diatomic Molecule



b = equilibrium distance between particles

$$V = \frac{1}{2}k(x_2 - x_1 - b)^2,$$

Define $\eta_i = x_i - x_{0i}$ equilibrium positions,

$$x_{02} - x_{01} = b$$

$$V = \frac{1}{2}k(\eta_2 - \eta_1)^2$$

Matrix of second derivatives:

$$\underline{V} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$$

kinetic energy: $T = \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2)$

$$\underline{T} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

Secular equation: $\det(\underline{V} - \omega^2 \underline{T}) = 0$

$$\det\begin{pmatrix} k - \omega^2 m & -k \\ -k & k - \omega^2 m \end{pmatrix} = 0$$

$$m\omega^2(\omega^2 m - 2k) = 0$$

Solutions: $\boxed{\omega_1 = 0}$, $\boxed{\omega_2 = \sqrt{\frac{2k}{m}}}$

$\omega_1 = 0 \rightarrow \ddot{\eta}_2 - \ddot{\eta}_1 = 0$ — uniform translational motion

$\ddot{\zeta}_1 = 0$

Eigenvectors normalized by $\vec{a}^T \underline{T} \vec{a} = \mathbb{1}$.

$$\boxed{(a_1)_1 = (a_1)_2 = \frac{1}{\sqrt{2m}}}$$

$$\omega_2 = \sqrt{\frac{2k}{m}} \Rightarrow \boxed{(a_2)_1 = -(a_2)_2 = \frac{1}{\sqrt{2m}}}$$

$$\underline{A} = \begin{pmatrix} \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} & -\frac{1}{\sqrt{2m}} \end{pmatrix}$$

Matrix of eigenvectors

Normal mode motion:

$\omega_1 = 0 \qquad \rightarrow \qquad \rightarrow$

$\omega_2 \qquad \leftarrow \qquad \rightarrow$

Normal coordinates ζ_k such that $T = \frac{1}{2}\sum \dot{\zeta}_k \dot{\zeta}_k$

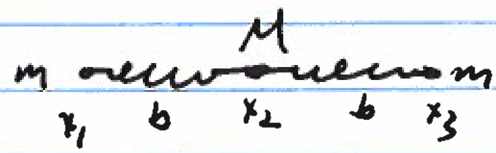
$V = \frac{1}{2}\sum (\omega_k^2) \zeta_k^2$

$$\vec{\eta} = \underline{A}\vec{\zeta} \xrightarrow{\text{Invert}} \vec{\zeta} = \underline{A}^{-1}\vec{\eta}$$

$$\underline{A}^{-1} = \sqrt{\frac{m}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\zeta_1 = \sqrt{\frac{m}{2}}(\eta_1 + \eta_2), \quad \zeta_2 = \sqrt{\frac{m}{2}}(\eta_1 - \eta_2)$$

Example: Linear Triatomic molecule



b = equilibrium distance between particles.

$$V = \frac{k}{2}(x_2 - x_1 - b)^2 + \frac{k}{2}(x_3 - x_2 - b)^2$$

Define $\eta_i = x_i - x_{0i}$
$\quad\uparrow$ equilibrium position

$$x_{02} - x_{01} = b = x_{03} - x_{02}$$

$$V = \frac{k}{2}(\eta_2 - \eta_1)^2 + \frac{k}{2}(\eta_3 - \eta_2)^2$$

$$V = \frac{k}{2}\left(\eta_1^2 + 2\eta_2^2 + \eta_3^2 - 2\eta_1\eta_2 - 2\eta_2\eta_3\right)$$

Matrix of second derivatives:

$$\underline{V} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

Kinetic energy: $T = \frac{1}{2}m(\dot\eta_1^2 + \dot\eta_3^2) + \frac{1}{2}M\dot\eta_2^2$

$$\underline{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

Secular equation! $\quad \det(\underline{V} - \omega^2 \underline{I}) = 0$

$$\det\begin{pmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{pmatrix} = 0$$

Can be written (Exercise):

$$\boxed{\omega^2\left(k^2-\omega^2 m\right)\left(k(M+2m)-\omega^2 M m\right)=0}$$

Solutions! $\omega_1 = 0$, $\omega_2 = \sqrt{\frac{k}{m}}$, $\omega_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}$

$\omega_1 = 0 \rightarrow \ddot{\zeta}_1 = 0$ — uniform translational motion

Eigenvector $(a_1)_1 = (a_1)_2 = (a_1)_3$.

(– Can eliminate this solution by fixing center of mass to origin:

$m(x_1+x_3) + Mx_2 = 0$.)

$\omega_2 = \sqrt{\frac{k}{m}} \rightarrow (a_2)_1 = -(a_2)_3, \quad (a_2)_2 = 0.$

$$\vec{a}_2 = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

with normalization $\boxed{m\left((a_n)_1^2 + (a_n)_3^2\right) + M(a_n)_2^2 = 1}$

$$\omega_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}$$

Eigenvector (exercise):

$$(q_3)_1 = \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}}, \quad (q_3)_2 = \frac{-2}{\sqrt{2M\left(2+\frac{M}{m}\right)}}$$

$$(q_3)_3 = \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}}$$

Normal mode motion:

ω_1 $\quad \rightarrow \quad \rightarrow \quad \rightarrow$

ω_2 $\quad \leftarrow \quad \bullet \quad \rightarrow$

ω_3 $\quad \leftarrow \quad \rightarrow \quad \leftarrow$

Normal coordinates: Invert $\vec{\eta} = \underline{A}\,\vec{\zeta} \rightarrow \vec{\zeta} = \underline{A}^{-1}\vec{\eta}$.

Exercise $\Longrightarrow$

$$\zeta_1 = \frac{1}{\sqrt{2m+M}}\left(m\eta_1 + M\eta_2 + m\eta_3\right)$$

$$\zeta_2 = \sqrt{\frac{m}{2}}\left(\eta_1 - \eta_3\right)$$

$$\zeta_3 = \sqrt{\frac{mM}{2(2m+M)}}\left[(\eta_1+\eta_3) - 2\eta_2\right]$$