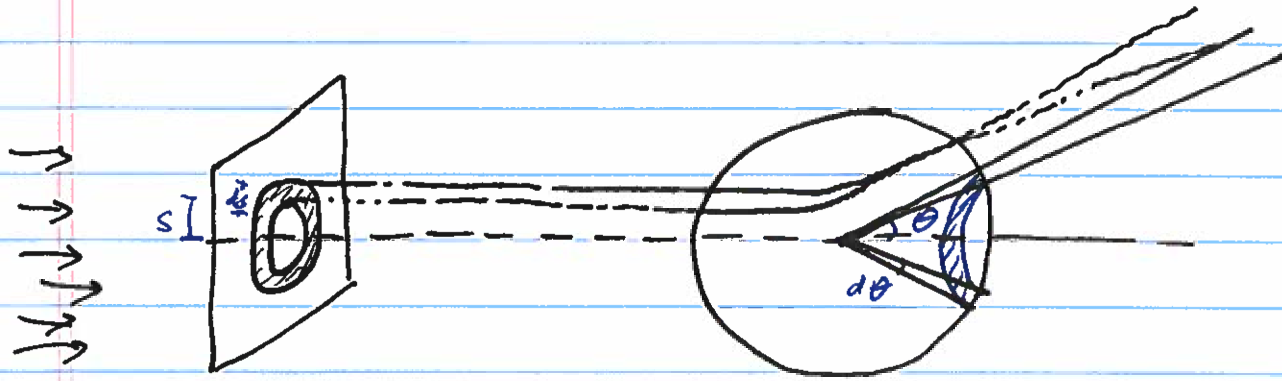
Review: Scattering, Rigid Body Motion, Fictitious Forces.

Scattering



Differential Scattering Cross section $\sigma(\Omega)$

$$\sigma(\Omega)\, d\Omega = \frac{\#\text{ particles scattered into solid angle } d\Omega \text{ / unit time}}{\#\text{ particles in beam/unit area/unit time}}$$

$\sigma(\Omega)$ has units of area — heuristically measures effective size of scattering source.

Central force: $\sigma(\Omega)$ is symmetric about axis through scattering center: $d\Omega = 2\pi \sin\theta\, d\theta$

$$\sigma(\Omega)\, d\Omega = 2\pi \sin\theta\, \sigma(\theta)\, d\theta$$

$$\boxed{\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|}$$

, s = function of beam energy E, and θ. (Impact parameter)

(really, θ = function of E, s.)

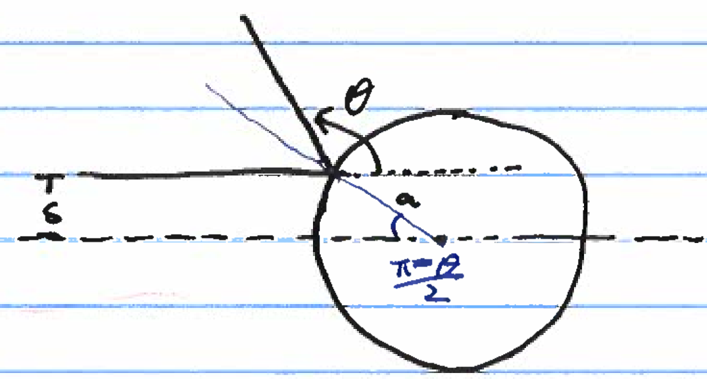
Example: Rutherford Cross section

Analysis of trajectories in potential $V = \frac{ZZ'e^2}{r}$

gives $s = \frac{ZZ'e^2}{2E} \cot\left(\frac{\theta}{2}\right)$

$$\sigma(\theta) = \frac{1}{4}\left(\frac{ZZ'e^2}{2E}\right)^2 \csc^4\left(\frac{\theta}{2}\right)$$

Example: Hard sphere scattering, sphere of radius a



$$s = a \sin\left(\frac{\pi-\theta}{2}\right) = a \cos\frac{\theta}{2}$$

$$\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right| = \frac{a\cos\theta/2}{\sin\theta} \cdot \frac{a}{2}\sin\frac{\theta}{2}$$

$$= \frac{a^2}{4}$$

$$\sigma = \int_0^\pi \sigma(\theta) \cdot 2\pi \sin\theta \, d\theta = -2\pi \frac{a^2}{4} \cos\theta \Big|_0^\pi$$

$$= \pi a^2$$

= cross-sectional area of sphere.

Orthogonal Transformations

If the components of a vector $\vec{V}$ transform as
$\vec{V}' = \underline{A}\,\vec{V}$ for some 3×3 matrix $\underline{A}$,
then $|\vec{V}'| = |\vec{V}|$ for all $\vec{V}$, if

$$(\underline{A}\vec{v})^T(\underline{A}\vec{v}) = \vec{v}^T\underline{A}^T\underline{A}\,\vec{v} = \vec{v}^T\vec{v}$$

$$\rightarrow \boxed{\underline{A}^T\underline{A} = \mathbb{1}}, \text{ or } \underline{A}^T = \underline{A}^{-1}$$

$\det \underline{A} = +1 \longrightarrow$ Proper orthogonal transformation

$\det \underline{A} = -1 \longrightarrow$ Improper orthogonal transformation — not smoothly connected to $\mathbb{1}$.

Eigenvalues of proper orthogonal $\underline{A}$ are $1, e^{i\Phi}, e^{-i\Phi}$

By a similarity transformation, $\underline{A}$ can be put in the form,

$$\underline{S}^{-1}\underline{A}\,\underline{S} = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a rotation about the z-axis.

Euler's Theorem: The general displacement of a rigid body with one point fixed is a rotation about some axis.

Euler Angles - Describe orientation by 3 successive rotations from space axes to body axes

1) Rotate about z-axis by ϕ
2) Rotate about new x-axis by θ
3) Rotate about new z-axis by ψ

Multiplying the three rotation matrices gives:

$$\underline{A} = \begin{pmatrix} c_\psi c_\phi - c_\theta s_\phi s_\psi & c_\psi s_\phi + c_\theta c_\phi s_\psi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\theta s_\phi c_\psi & -s_\psi s_\phi + c_\theta c_\phi c_\psi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{pmatrix}$$

where $c_\psi = \cos\psi$, $s_\psi = \sin\psi$, etc.

$\vec{x}' = \underline{A}\vec{x}$, where $\vec{x}'$ is expressed in the body frame, $\vec{x}$ is expressed in the space frame.

To convert from body frame to space frame, transpose A:

$$\vec{x} = \underline{A}^{-1}\vec{x}' = \underline{A}^T\vec{x}'$$ because $\underline{A}$ is orthogonal.

Rate of change of a vector

$\left(\frac{d\vec{a}}{dt}\right)_{space}$ — rate of change of $\vec{a}$ relative to space frame (inertial)

$\left(\frac{d\vec{a}}{dt}\right)_{body}$ — rate of change of $\vec{a}$ relative to body frame, fixed relative to body (noninertial)

$$\boxed{\left(\frac{d\vec{a}}{dt}\right)_{space} = \left(\frac{d\vec{a}}{dt}\right)_{body} + \vec{\omega} \times \vec{a}}$$

instantaneous angular velocity of body relative to space.

In the body frame, in terms of the Euler angles,

$$\vec{\omega} = \begin{pmatrix} \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \dot{\phi} \cos\theta + \dot{\psi} \end{pmatrix}$$

In the space frame,

$$\vec{\omega} = \begin{pmatrix} \dot{\theta} \cos\phi + \dot{\psi} \sin\theta \sin\phi \\ \dot{\theta} \sin\phi - \dot{\psi} \sin\theta \cos\phi \\ \dot{\psi} \cos\theta + \dot{\phi} \end{pmatrix}$$

Coriolis and Centrifugal Effects (Fictitious forces)

Consider a uniformly rotating frame w/ angular velocity $\vec{\omega}$ relative to the inertial space frame.

$$\vec{V}_s = \left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega}\times\vec{r} = \vec{V}_r + \vec{\omega}\times\vec{r}$$

$$\vec{a}_s = \left(\frac{d^2\vec{r}}{dt^2}\right)_s = \frac{d}{dt}\left(\vec{V}_r + \vec{\omega}\times\vec{r}\right)_r + \vec{\omega}\times\left(\vec{V}_r + \vec{\omega}\times\vec{r}\right)$$

$$= \left(\frac{d\vec{V}_r}{dt}\right)_r + 2\left(\vec{\omega}\times\vec{V}_r\right) + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$$

$$\equiv \vec{a}_r + 2\left(\vec{\omega}\times\vec{V}_r\right) + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$$

$$\vec{F} = m\vec{a}_s \longrightarrow m\vec{a}_r = \vec{F} - \underbrace{2m\left(\vec{\omega}\times\vec{V}_r\right)}_{\text{Coriolis effect}} - \underbrace{m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)}_{\text{Centrifugal effect}}$$

We may use Newton's 2^{nd} Law in the rotating (noninertial) frame if we add fictitious forces proportional to m to the external force $\vec{F}$.

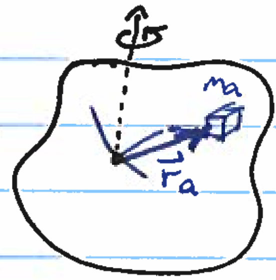
Inertia Tensor

$\vec{L} = \underline{I}\,\vec{\omega}$, or $L_i = I_{ij}\,\omega_j$

angular momentum   Inertia tensor   angular velocity

$$\boxed{I_{ij} = \sum_{\substack{\text{particles}\\ a}} m_a \left(r_a^2 \delta_{ij} - r_{a_i} r_{a_j} \right)}$$

Inertia tensor for motion w/ one point (the origin) fixed.



Continuous medium:

$$\boxed{I_{ij} = \int_V dV\, \rho(\vec{r}) \left(r^2 \delta_{ij} - x_i x_j \right)}$$

Under an orthogonal transformation:

$$\underline{I}' = \underline{A}\,\underline{I}\,\underline{A}^T \quad , \text{or} \quad I'_{ij} = a_{ik}\, a_{jl}\, I_{kl}$$

— transformation law for rank-2 tensor under rotations.

Kinetic Energy: $T = \frac{1}{2} \vec{\omega} \cdot \underline{I} \vec{\omega}$

Define $\vec{\omega} = \omega \hat{n}$.

$$T = \frac{1}{2} (\hat{n} \cdot \underline{I} \hat{n}) \omega^2 = \frac{1}{2} I \omega^2$$

$I = \hat{n} \cdot \underline{I} \hat{n}$ = moment of inertia about axis of rotation.

Principal Axes and Principal Moments of Inertia

$\underline{I}$ can be diagonalized by a similarity transformation by an orthogonal matrix $\underline{A}$:

$$\underline{I}_D = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = \underline{A} \, \underline{I} \, \underline{A}^T \quad \text{for some orthogonal } \underline{A}.$$

I_1, I_2, I_3 — principal moments of inertia = eigenvalues of $\underline{I}$.

Principal axes = eigenvectors of $\underline{I}$.

In principal axis frame, $\vec{L} = \underline{I} \vec{\omega}$ becomes

$$L_1 = I_1 \omega_1, \quad L_2 = I_2 \omega_2, \quad L_3 = I_3 \omega_3$$

Euler's Equations of motion:

In an inertial frame w/ origin at the fixed pt. of the rigid body,

$$\left(\frac{d\vec{L}}{dt}\right)_s = \vec{N} \quad \text{torque}$$

In terms of the body frame, $\left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_b + \vec{\omega} \times \vec{L}$

Choose the principal axis frame in the body, so that $L_1 = I_1 \omega_1$, etc.

$$\Rightarrow \left.\begin{aligned} I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) &= N_1 \\ I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) &= N_2 \\ I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) &= N_3 \end{aligned}\right\} \text{ Euler's equations of motion.}$$

Torque-free motion: $N_1 = N_2 = N_3 = 0$.
Symmetric body: suppose $I_1 = I_2$. Then $\dot{\omega}_3 = 0$.
— Motion generally precesses about ω_3 axis in body frame.

With torque: Motion is more complicated.
Symmetric top w/ $I_1 = I_2$, torque due to gravity
— precession + nutation (bobbing) of figure axis.