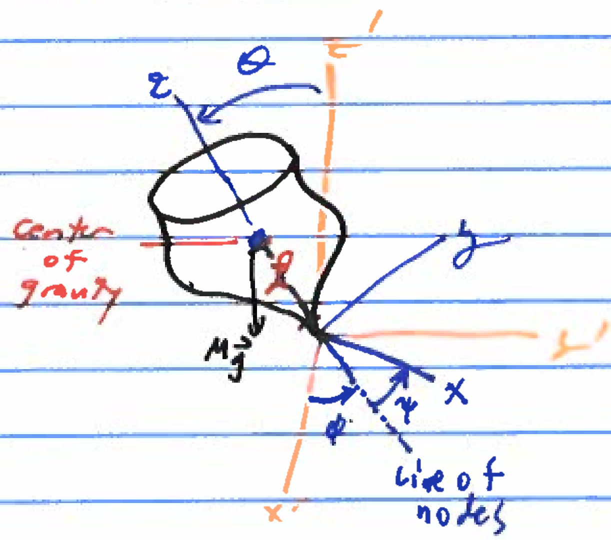
Symmetrical Top with one pt. fixed — Goldstein 5.7



ℓ = distance from center of gravity to fixed pt.

$\dot{\psi}$ = rotation of top about its figure axis z.
$\dot{\phi}$ = precession of figure axis about vertical axis z'
$\dot{\theta}$ = nutation (bobbing) of figure axis relative to the vertical axis z'.

Suppose $\dot{\psi} \gg \dot{\theta} \gg \dot{\phi}$, $I_1 = I_2 \neq I_3$.

Euler's Eqs of Motion:

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - \overset{I_1 = I_2}{I_1}) = N_1$$
$$I_2 \dot{\omega}_2 + \omega_1 \omega_3 (I_1 - I_3) = N_2$$
$$I_3 \dot{\omega}_3 = N_3$$

If initially $N_3 = 0 = N_2$, $N_1 \neq 0$, $\omega_1 = \omega_2 = 0$, $\omega_3 \neq 0$

$\Rightarrow \dot{\omega}_3 = 0$

Initially $\dot{\omega}_1 = N_1 / I_1 \neq 0 \rightarrow \omega_1$ changes

$\rightarrow \dot{\omega}_2 \neq 0 \rightarrow \omega_2$ changes.

It is easier to visualize this in terms of the Euler 4's.

Lagrangian: $T = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$

Body frame

Recall: (Lecture 13)

$$\omega_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$
$$\omega_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$
$$\omega_3 = \dot\phi \cos\theta + \dot\psi$$

$$\omega_1^2 + \omega_2^2 = \dot\theta^2 + \dot\phi^2 \sin^2\theta$$

$$\Rightarrow T = \frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi \cos\theta)^2$$

potential energy: $V = \sum_a -m_a \vec{r}_a \cdot \vec{g}$

$$= -\underset{\text{total mass}}{M} \underset{\text{center of mass}}{\vec{R}} \cdot \vec{g} = Mg\ell\cos\theta$$

$$L = \frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 - Mg\ell\cos\theta$$

ϕ and ψ are cyclic coordinates

$$\rightarrow \begin{cases} P_\psi = \dfrac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = I_3 \omega_3 \quad \text{conserved} \\ P_\phi = \dfrac{\partial L}{\partial \dot\phi} = I_1 \dot\phi \sin^2\theta + I_3 (\dot\psi + \dot\phi\cos\theta)\cos\theta \quad \text{conserved.} \end{cases}$$

Energy $E = T + V = \frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 \omega_3^2 + Mg\ell\cos\theta$ conserved.

Define a, b s.t. $P_\psi = I_1 a$, $P_\phi = I_1 b$

$$I_3 \dot\psi = I_1 a - I_3 \dot\phi \cos\theta$$

$$\rightarrow I_1 \dot\phi \sin^2\theta + I_1 a \cos\theta = I_1 b$$

$$\boxed{\dot\phi = \frac{b - a\cos\theta}{\sin^2\theta}}$$

Then, $\boxed{\dot\psi = \frac{I_1 a}{I_3} - \cos\theta \, \frac{b - a\cos\theta}{\sin^2\theta}}$

If we knew $\theta(t)$, we could solve for $\phi(t)$ and $\psi(t)$.

$P_\psi = I_3 \omega_3 = I_1 a \longrightarrow \omega_3 = \frac{I_1 a}{I_3}$ constant.

Define $E' = E - \frac{1}{2} I_3 \omega_3^2$ constant

$$= \frac{1}{2} I_1 \dot\theta^2 + \frac{1}{2} I_1 \frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

— Like an effective one-dim'l problem with effective potential

$$V_{eff}(\theta) = Mgl\cos\theta + \frac{1}{2} I_1 \left(\frac{b - a\cos\theta}{\sin\theta}\right)^2$$

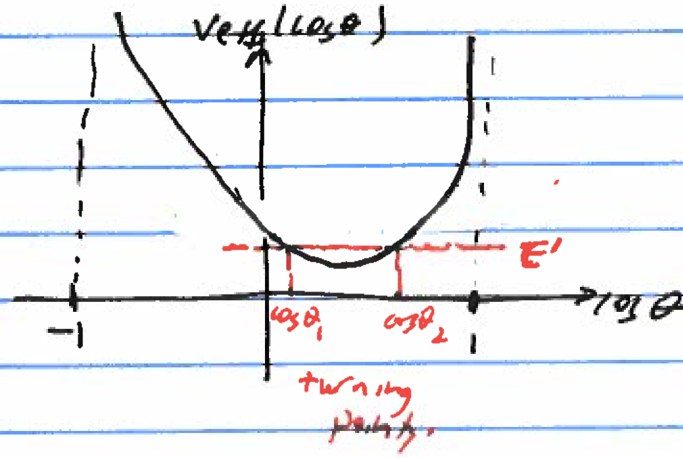
Think of V_{eff} as a function of $\cos\theta$:

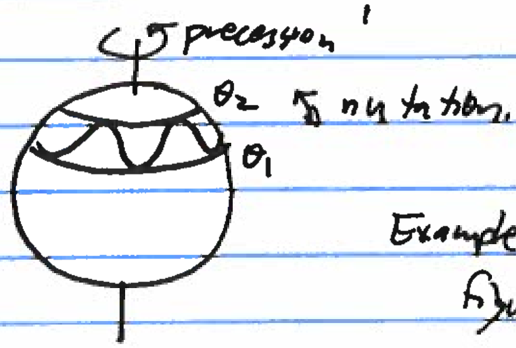
$$V_{eff}(\cos\theta) = Mgl\cos\theta + \frac{1}{2} I_1 \left(\frac{b - a\cos\theta}{\sqrt{1-\cos^2\theta}} \right)^2$$

$$= Mgl\cos\theta + \frac{I_1}{2(1-\cos^2\theta)} (b - a\cos\theta)^2$$

$(1-\cos^2\theta)\, V_{eff}(\cos\theta)$ is cubic in $\cos\theta$.

$V_{eff}(\cos\theta)$ diverges as $\cos\theta \to 1$





Example of motion of the top's figure axis.