Rigid Body Equations of Motion: Goldstein Ch 5

Angular Momentum and Kinetic Energy about a point

In many physical systems of interest, the translational and rotational motion can be analyzed separately, i.e. the equations of motion for coordinates describing the position, e.g. of the center of mass, and the equations of motion for the coordinates describing the orientation of a body, separate.

We have seen (c.f. lecture 1) that the angular momentum can be written as a sum of terms, one depending on the total mass M, center of mass position $\vec{R}$, and center of mass velocity $\vec{v}$; and the other depending on the position of the individual component masses $\vec{r}_a{}'$ and momenta $\vec{p}_a{}'$, relative to the center of mass.

$$\vec{L} = M\vec{R}\times\vec{v} + \sum_a \vec{r}_a{}'\times\vec{p}_a{}' \quad , \quad a = 1, \ldots, \#\text{particles}$$

Similarly, the kinetic energy $T = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_i m_i v_i^2$ can be written as

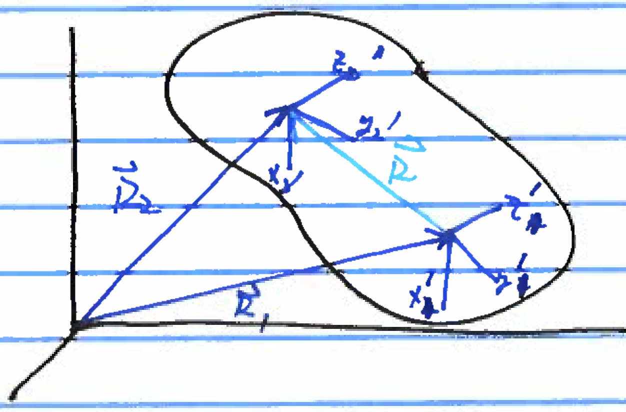
$$T = \underbrace{\tfrac{1}{2}Mv^2}_{\text{center of mass motion}} + \underbrace{T'(\phi,\theta,\psi)}_{\text{motion about center of mass.}}$$

Similarly, the potential energy function often depends only on the translational coordinates (e.g. gravity) or rotational coordinates (e.g. magnetic field $\vec{B}$ acting on magnetic dipole $\vec{M}$, with $V = -\vec{\mu}\cdot\vec{B}$).

Then, $L = T - V$ gives rise to separate equations of motion for the translational and rotational degrees of freedom.

We will need expressions for the angular momentum and kinetic energy about some point fixed in the body.

Note that the rotation angle of the rigid body displacement, and the instantaneous $\not\triangleleft$ velocity vector, are independent of the origin of the body-fixed axes:



$\vec{R} = \vec{R}_2 - \vec{R}_1$

$\vec{R}_2$ defined relative to $\vec{R}_1$:

$\swarrow$ velocity in x_1', y_1', z_1' frame

$$\left(\frac{d\vec{R}_2}{dt}\right)_s = \left(\frac{d\vec{R}_1}{dt}\right)_s + \left(\frac{d\vec{R}}{dt}\right)_s = \left(\frac{d\vec{R}_1}{dt}\right)_s + \vec{\omega}_1 \times \vec{R}$$

$\vec{R}_1$ defined relative to $\vec{R}_2$:

$\swarrow$ velocity in x_2', y_2', z_2' frame.

$$\left(\frac{d\vec{R}_1}{dt}\right)_s = \left(\frac{d\vec{R}_2}{dt}\right)_s - \left(\frac{d\vec{R}}{dt}\right)_s = \left(\frac{d\vec{R}_2}{dt}\right)_s - \vec{\omega}_2 \times \vec{R}$$

Comparing, $\left(\frac{d\vec{R}_2}{dt}\right)_s - \left(\frac{d\vec{R}_1}{dt}\right)_s = \vec{\omega}_1 \times \vec{R} = \vec{\omega}_2 \times \vec{R}$

$$\Rightarrow (\vec{\omega}_2 - \vec{\omega}_1) \times \vec{R} = \vec{0}$$

$\rightarrow$ $\vec{\omega}_2 - \vec{\omega}_1$ lies along line connecting points 1, 2.

This is true for all points 1, 2.
For $\vec{\omega}$ to be a nonsingular vector field, $\vec{\omega}_2 = \vec{\omega}_1$

$\Rightarrow$ $\boxed{\vec{\omega} \text{ is independent of the origin of the body-fixed axes.}}$ ∎

If we consider rotation with one pt. fixed, define $\vec{r}_a$, $\vec{v}_a$ radius vector and velocity of a^{th} particle relative to the fixed point.

$$\vec{L} = \sum_a m_a (\vec{r}_a \times \vec{v}_a)$$

$$= \sum_a m_a \vec{r}_a \times (\vec{\omega} \times \vec{r}_a)$$

$$= \sum_a m_a \left[\vec{\omega}\, r_a^2 - \vec{r}_a (\vec{r}_a \cdot \vec{\omega})\right]$$

Example: $L_x = \sum_a m_a \left[\omega_x (r_a^2 - x_a^2) - x_a y_a \omega_y - x_a z_a \omega_z\right]$

$$\equiv I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$I_{xx} = \sum_a m_a (r_a^2 - x_a^2)$$

$$I_{xy} = -\sum_a m_a x_a y_a$$

$$I_{xz} = -\sum_a m_a x_a z_a$$

Similarly for $L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

I_{ij} are the components of the (moment of) inertia tensor.

For continuous bodies:

$$I_{xx} = \int_V \rho(\vec{r})(r^2 - x^2)\, dV, \text{ etc.}$$

Generally, $\boxed{I_{jk} = \int_V \rho(\vec{r})(r^2 \delta_{jk} - x_j x_k)\, dV}$

Under a transformation $x_i' = a_{ij} x_j$,

$$\boxed{I'_{jk} = a_{jl} a_{km} I_{lm}}$$ ← Transformation law for 2nd rank tensor.

As a matrix equation,

$$\boxed{\underline{I}' = A \underline{I} A^T = A \underline{I} A^{-1}}$$ for orthogonal matrix A.

In the transformed frame the angular momentum vector is $\vec{L}' = A\vec{L}$, and the angular velocity vector is $\vec{\omega}' = A\vec{\omega}$.

We can check consistency of these transformation rules:

$$\vec{L} = \underline{I}\,\vec{\omega}. \Rightarrow \vec{L}' = (A\underline{I}A^{-1})(A\vec{\omega}) = A(\underline{I}\vec{\omega}) = A\vec{L} \checkmark.$$

We can write the kinetic energy of motion about a pt. in terms of $\vec{\omega}$, $\underline{I}$:

$$T = \frac{1}{2}\sum_a m_a v_a^2 = \frac{1}{2}\sum_a m_a \vec{v}_a \cdot (\vec{\omega} \times \vec{r}_a)$$

$$= \frac{1}{2}\sum_a \vec{\omega} \cdot [m_a(\vec{r}_a \times \vec{v}_a)]$$

$$= \frac{1}{2}\vec{\omega} \cdot \vec{L}$$

$$\boxed{T = \frac{1}{2}\vec{\omega} \cdot (\underline{I}\,\vec{\omega})}$$

Define $\vec{\omega} = \omega\hat{n}$. Then, $T = \frac{1}{2} I \omega^2$, where

$$I = \hat{n} \cdot (\underline{I}\,\hat{n}) = \sum_a m_a\left[r_a^2 - (\vec{r}_a \cdot \hat{n})^2\right]$$

Moment of inertia about the axis of rotation.

Using $(\vec{r}\times\hat{n})_i = \epsilon_{ijk}\, x_j n_k$, and

$$\boxed{\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}},$$

$$(\vec{r}\times\hat{n})\cdot(\vec{r}\times\hat{n}) = (\epsilon_{ijk}\, x_j n_k)(\epsilon_{ilm}\, x_l n_m)$$

$$= (\delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl})\, x_j n_k x_l n_m$$

$$= (x_j x_j)(n_k n_k) - (x_j n_j)(x_k n_k)$$

$$= r^2 \cdot 1 - (\vec{r}\cdot\hat{n})^2$$

$$\Rightarrow I = \sum_a m_a \left[r_a^2 - (\vec{r}_a\cdot\hat{n})^2 \right]$$

$$= \sum_a m_a (\vec{r}_a\times\hat{n})\cdot(\vec{r}_a\times\hat{n})$$

$$= \sum_a \frac{m_a}{\omega^2} (\vec{r}_a\times\vec{\omega})\cdot(\vec{r}_a\times\vec{\omega})$$

$$= \sum_a \frac{m_a}{\omega^2} V_a^2$$

$\vec{v}_a = \vec{\omega}\times\vec{r}_a$ space-frame velocity

$$= \frac{2T}{\omega^2}$$

$$\Rightarrow \boxed{T = \frac{1}{2} I \omega^2}$$

We have,

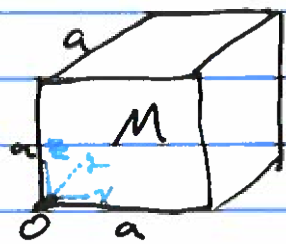
$$T_{rotation} = \frac{1}{2} I_{ij}\, \omega_i \omega_j \ , \quad \text{where}$$

kinetic energy of rotation about some axis

$$I_{ij} = \sum_a m_a \left(\delta_{ij} r_a^{\,2} - (x_a)_i (x_a)_j \right)$$

continuous $\Rightarrow$ $I_{ij} = \int_V \rho(\vec{x}) \left(\delta_{ij} r^2 - x_i x_j \right) dV$, as before.

Example:



Cube of uniform density

$$\rho = \frac{M}{a^3}$$

Moment of Inertia tensor about O:

$$I_{xx} = \int_V \frac{M}{a^3} \left((x^2+y^2+z^2) - x^2 \right) dV$$

$$= \int_0^a \int_0^a \int_0^a dx\, dy\, dz \ \frac{M}{a^3} (y^2+z^2)$$

$$= \frac{M}{a^3} \cdot a \cdot \left(\frac{a^3}{3} \cdot a + \frac{a^3}{3} \cdot a \right) = \frac{2}{3} M a^2$$

Similarly,

$$\underline{\underline{I}} = M a^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$$

Principal Axes, Eigenvalues of Inertia Tensor

note $I_{ij} = I_{ji}$ (symmetric matrix)

$\to$ 6 independent components.

I_{ij} depends on location of origin of body, and orientation of body axes.

$\exists$ coordinates such that $\underline{I}$ is diagonal, and

$$\vec{L} = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix} \vec{\omega}$$

, e.g. $L_1 = I_1 \omega_1$, $L_2 = I_2 \omega_2$, $L_3 = I_3 \omega_3$.

I_1, I_2, I_3 — "principal moments of inertia" = eigenvalues of $\underline{I}$.

Define $$\underline{I}_D = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix} = A \underline{I} A^T$$ for some orthogonal matrix A.

The axes rotated by A are the "principal axes" = eigenvectors of $\underline{I}$.