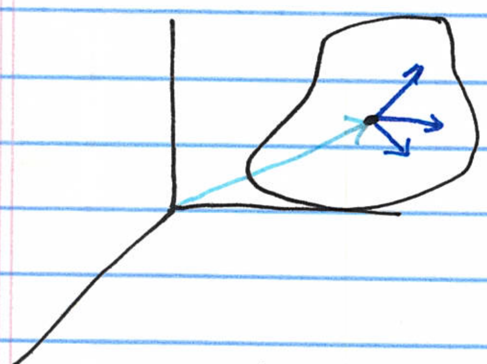
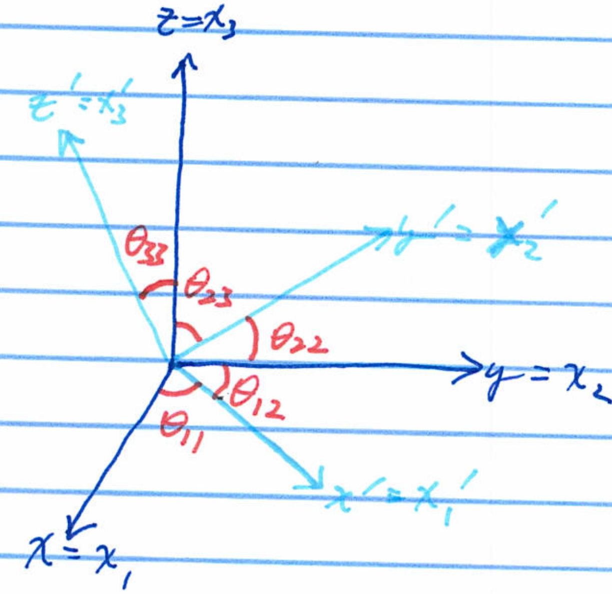
The Kinematics of Rigid Body Motion

In a rigid body, the distances between matter points are treated as fixed.
There are $\frac{N(N-1)}{2}$ pairs of points with fixed separation, if there are N particles in the body. This is an overcomplete set of constraints on the $3N$ coordinates required to specify the positions of the N particles in the rigid body.



Intuitively, we need 6 generalized coordinates to specify a configuration of a rigid body: 3 to specify the location of a pt. in the body (e.g. the center of mass) and 3 to specify the orientation of the body about that point.

A convenient overcomplete set of generalized coordinates describing the orientation of the body is the direction cosines between coordinate axes fixed to the body and some reference coordinate axes.



$$\cos\theta_{11} = \hat{i}' \cdot \hat{i}$$
$$\cos\theta_{12} = \hat{i}' \cdot \hat{j}$$
$$\cos\theta_{21} = \hat{j}' \cdot \hat{i}$$
$$\cos\theta_{32} = \hat{k}' \cdot \hat{j}$$
$$\vdots$$
etc.

$\hat{i}$ = unit vector in x-direction
$\hat{j}$ = " " " y-direction
$\hat{k}$ = " " " z-direction

similarly for $\hat{i}'$, $\hat{j}'$, $\hat{k}'$ and the x', y', z' directions

We can write the primed basis vectors in terms of the unprimed basis vectors using the direction cosines:

$$\hat{i}' = \cos\theta_{11}\,\hat{i} + \cos\theta_{12}\,\hat{j} + \cos\theta_{13}\,\hat{k}$$
$$\hat{j}' = \cos\theta_{21}\,\hat{i} + \cos\theta_{22}\,\hat{j} + \cos\theta_{23}\,\hat{k}$$
$$\hat{k}' = \cos\theta_{31}\,\hat{i} + \cos\theta_{32}\,\hat{j} + \cos\theta_{33}\,\hat{k}$$

Similarly, for position vector $\vec{r}$,

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \equiv x'\,\hat{i}' + y'\,\hat{j}' + z'\,\hat{k}'$$

$$x' = \vec{r}\cdot\hat{i}' = \cos\theta_{11}\,x + \cos\theta_{12}\,y + \cos\theta_{13}\,z$$
$$y' = \vec{r}\cdot\hat{j}' = \cos\theta_{21}\,x + \cos\theta_{22}\,y + \cos\theta_{23}\,z$$
$$z' = \vec{r}\cdot\hat{k}' = \cos\theta_{31}\,x + \cos\theta_{32}\,y + \cos\theta_{33}\,z$$

There are $\boxed{\text{9 direction cosines}}$, but only 3 coords. are needed to specify an orientation.

Relations: $\hat{i}'\cdot\hat{j}' = \hat{j}'\cdot\hat{k}' = \hat{k}'\cdot\hat{i}' = 0 = \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i}$

$\hat{i}'\cdot\hat{i}' = \hat{j}'\cdot\hat{j}' = \hat{k}'\cdot\hat{k}' = 1 = \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k}$

$$\hat{i}'\cdot\hat{j}' = 0 \rightarrow \cos\theta_{11}\cos\theta_{21} + \cos\theta_{12}\cos\theta_{22} + \cos\theta_{13}\cos\theta_{23} = 0$$

Generally, $\boxed{\sum_{\ell=1}^{3} \cos\theta_{m'\ell}\cos\theta_{m\ell} = 0}$, $m \neq m'$ $\leftarrow \boxed{\text{3 relations}}$

$$\hat{i}'\cdot\hat{i}' = 1 \rightarrow \cos^2\theta_{11} + \cos^2\theta_{12} + \cos^2\theta_{13} = 1$$

Generally, $\boxed{\sum_{\ell=1}^{3} \cos^2\theta_{m\ell} = 1}$ $\leftarrow \boxed{\text{3 relations}}$

Orthogonal Transformations

For notational simplicity, define $a_{ij} \equiv \cos\theta_{ij}$.

Transformations between coordinates are a linear transformation:

$$x'_i = \sum_{j=1}^{3} a_{ij} x_j \equiv a_{ij} x_j$$

Summation Convention - Repeated indices are summed over.

$$x_i x_i = \sum_i (x_i)^2$$

A vector keeps its magnitude when coordinates are transformed:

$$x_i' x_i' = x_i x_i$$

$$\Rightarrow (a_{ij} x_j)(a_{ik} x_k) = x_i x_i = \delta_{jk} x_j x_k$$

Kronecker delta $\delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$

$$\Rightarrow \boxed{a_{ij} a_{ik} = \delta_{jk}}, \quad j, k = 1, 2, 3$$

Collect the transformation elements a_{ij} in a matrix:

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A^T A = \mathbb{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(A^T)_{ji} = A_{ij}$

transpose - Denoted $\tilde{A}$ in Goldstein.

Successive transformations → matrix multiplication

$$x_k' = b_{kj}\, x_j \quad , \quad x_i'' = a_{ik}\, x_k'$$

$$\Rightarrow x_i'' = a_{ik}\, b_{kj}\, x_j \equiv c_{ij}\, x_j$$

$$c_{ij} = a_{ik}\, b_{kj} \Leftrightarrow C = AB$$

Suppose $A^T A = B^T B = \mathbb{1}$ (i.e. A and B are <u>orthogonal</u> matrices)

Then $C^T C = (AB)^T (AB) = B^T A^T A B$

$$= B^T \mathbb{1} B = B^T B$$

$$= \mathbb{1}.$$

(i.e. C is also an orthogonal matrix.)

Note that in general, $A(BC) = (AB)C$

(associativity)

But $AB \neq BA$ (noncommutativity)

Consider the determinant, recalling $\det A^T = \det A$,

$\det AB = (\det A)(\det B)$

Then, $\det(A^T A) = (\det A)^2 = \det(\mathbb{1}) = 1$

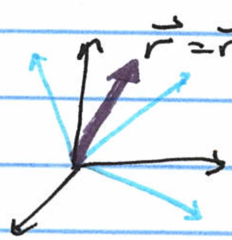
If $A^T A = \mathbb{1}$

$$\Rightarrow \boxed{\det A = \pm 1}$$

Active vs. Passive Transformations.

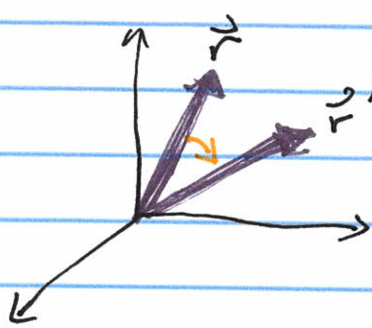
We can either consider vectors fixed and coordinates changing orientation, or hold coordinates fixed and change the orientation of vectors.

Vectors fixed $\quad \vec{r} \to A\vec{r} \equiv \vec{r}'$. Passive transformation



Alternatively, transform vector in opposite sense for equivalent transformation leaving coordinates fixed.

$$\vec{r} \to A^{-1}\vec{r} = \vec{r}' \qquad \text{Active transformation}$$



$$A^T A = \mathbb{1} \longrightarrow \underset{\text{transpose}}{A^T} = \underset{\text{inverse}}{A^{-1}}$$

$$A^{-1}A = AA^{-1} = \mathbb{1}$$

$\rightarrow$ $A^T A = AA^T = \mathbb{1}$ for orthogonal matrices.

$$AA^T = \mathbb{1} \longrightarrow A_{m\ell} A_{m'\ell} = \delta_{mm'}$$

With $A_{m\ell} = \cos\theta_{m\ell}$,

$$\boxed{\begin{array}{l} \sum_\ell \cos\theta_{m\ell} \cos\theta_{m'\ell} = 0 \quad , \quad m' \neq m \\ \sum_\ell (\cos\theta_{m\ell})^2 = 1 \end{array}}$$

These are the relations between direction cosines found earlier.