Phys 475 S'10 Problem Set 5 Solutions

3.11.33 $M = \begin{pmatrix} A & H \\ H & B \end{pmatrix}$, A, H, B real.

Eigenvalues λ satisfy $\det(M - \lambda \mathbb{1}) = 0$.

$$\begin{vmatrix} A-\lambda & H \\ H & B-\lambda \end{vmatrix} = (A-\lambda)(B-\lambda) - H^2 = 0$$

$$\lambda^2 - (A+B)\lambda + (AB - H^2) = 0$$

$$\lambda_\pm = \frac{A+B \pm \sqrt{(A+B)^2 - 4(AB-H^2)}}{2}$$

$$\lambda_\pm = \boxed{\frac{A+B}{2} \pm \frac{1}{2}\sqrt{(A-B)^2 + 4H^2}}$$

$\lambda_\pm$ is real because the argument of the square root, $(A-B)^2 + 4H^2 > 0$.

Eigenvectors:

$$\begin{pmatrix} A-\lambda & H \\ H & B-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

λ_+: $$\begin{pmatrix} \frac{A-B}{2} - \frac{1}{2}\sqrt{(A-B)^2+4H^2} & H \\ H & \frac{B-A}{2} - \frac{1}{2}\sqrt{(A-B)^2+4H^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left[\frac{A-B}{2} - \frac{1}{2}\sqrt{(A-B)^2+4H^2}\right] x + H y = 0$$

$\rightarrow$ Eigenvector (w/ $x=1$): $\vec{V}_+ = \begin{pmatrix} 1 \\ \frac{B-A}{2H} + \frac{1}{2H}\sqrt{(A-B)^2+4H^2} \end{pmatrix}$

Similarly, $\vec{V}_- = \begin{pmatrix} 1 \\ \frac{B-A}{2H} - \frac{1}{2H}\sqrt{(A-B)^2+4H^2} \end{pmatrix}$

$$\vec{V}_+ \cdot \vec{V}_- = 1 + \left(\frac{B-A}{2H}\right)^2 - \frac{1}{(2H)^2}\left[(A-B)^2 + 4H^2\right] = 0$$

Hence, $\vec{V}_+ \perp \vec{V}_-$. ✓

3.11.57 D diagonal $\rightarrow$ $D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_N \end{pmatrix}$

By direct calculation, $D^n = \begin{pmatrix} d_1^n & & & \\ & d_2^n & & \\ & & \ddots & \\ & & & d_N^n \end{pmatrix}$

If $D = C^{-1}MC$, then $D^n = (C^{-1}MC)^n$

$$= \underbrace{C^{-1}MC\,C^{-1}MC \cdots C^{-1}MC}_{n \text{ factors of } C^{-1}MC}$$

Since $CC^{-1} = \mathbb{1}$, $D^n = C^{-1}M\mathbb{1}M\mathbb{1}\cdots MC$

$$= C^{-1}M^nC$$

Multiplying by C on the left and C^{-1} on the right,

$$CD^nC^{-1} = M^n$$

3.12.14 $V = \frac{1}{2}kx^2 + \frac{1}{2}\cdot 2k(x-y)^2 + \frac{1}{2}ky^2$

$$m\ddot{x} = -m\omega^2 x = -\frac{\partial V}{\partial x} = -3kx + 2ky$$

$$m\ddot{y} = -m\omega^2 y = -\frac{\partial V}{\partial y} = 2kx - 3ky$$

$$\frac{m\omega^2}{k}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{vmatrix} 3-\frac{m\omega^2}{k} & -2 \\ -2 & 3-\frac{m\omega^2}{k} \end{vmatrix} = \left(3-\frac{m\omega^2}{k}\right)^2 - 4 = 0$$

$$\left(\frac{m\omega^2}{k}\right)^2 - 6\left(\frac{m\omega^2}{k}\right) + 5 = 0$$

$$\frac{m\omega^2}{k} = \frac{6 \pm \sqrt{36-20}}{2} = 3 \pm 2$$

$\boxed{\omega_+^2 = \frac{5k}{m}}$ $\rightarrow$ $(3-5)x - 2y = -2x - 2y = 0 \rightarrow y = -x$

Eigenvector $\boxed{\vec{v}_+ = (1,-1)}$. If $x = A\cos(\omega t)$ then $y = -A\cos(\omega t)$.

$\boxed{\omega_-^2 = \frac{k}{m}}$ $\rightarrow$ $(3-1)x - 2y = 2x - 2y = 0 \rightarrow y = x$

Eigenvector $\boxed{\vec{v}_- = (1,1)}$. If $x = A\cos(\omega t)$, $y = A\cos(\omega t)$.

(non-normalized)

10.5.6 a) $\delta_{ij}\delta_{jk}\delta_{km}\delta_{im} = \delta_{ik}\delta_{km}\delta_{im} = \delta_{im}\delta_{im} = \delta_{ii} = \boxed{3}$

b) $\epsilon_{ijk}\delta_{jk} = \frac{1}{2}(\epsilon_{ijk}\delta_{jk} + \epsilon_{ijk}\delta_{kj})$ using $\delta_{jk} = \delta_{kj}$

$= \frac{1}{2}(\epsilon_{ijk}\delta_{jk} + \epsilon_{ikj}\delta_{jk})$ exchanging dummy indices $j \leftrightarrow k$ in second term

$= \frac{1}{2}(\epsilon_{ijk}\delta_{jk} - \epsilon_{ijk}\delta_{jk})$ using $\epsilon_{ikj} = -\epsilon_{ijk}$

$= \boxed{0}$.

c) $\epsilon_{jk2}\epsilon_{k2j} = \epsilon_{k2j}\epsilon_{k2j} = \delta_{22}\delta_{jj} - \delta_{2j}\delta_{j2}$

$= 1\cdot 3 - \delta_{22} = 1\cdot 3 - 1 = \boxed{2}$

d) $\epsilon_{3jk}\epsilon_{kj3} = -\epsilon_{kj3}\epsilon_{kj3} = -(\delta_{jj}\delta_{33} - \delta_{j3}\delta_{3j})$

$= -3\cdot 1 + \delta_{33} = -3+1 = \boxed{-2}$

e) $\epsilon_{23i}\epsilon_{2i3} = -\epsilon_{i23}\epsilon_{i23} = -(\delta_{22}\delta_{33} - \delta_{23}\delta_{32})$

$= -(1\cdot 1 - 0\cdot 0) = \boxed{-1}$

f) $\epsilon_{k31}\epsilon_{3k1} = -\epsilon_{k31}\epsilon_{k31} = -(\delta_{33}\delta_{11} - \delta_{31}\delta_{13})$

$= -(1\cdot 1 - 0\cdot 0) = \boxed{-1}$

Additional Problems

1. $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\times\vec{B})_i(\vec{C}\times\vec{D})_i$

$= (\epsilon_{ijk}A_jB_k)(\epsilon_{ilm}C_lD_m)$

$= \epsilon_{ijk}\epsilon_{ilm}A_jB_kC_lD_m$

$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})A_jB_kC_lD_m$

$= A_jC_jB_kD_k - A_jD_jB_kC_k$

$= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$

2. a) $\det M = \epsilon_{ijk}\underset{\text{components of row 1}}{M_{1i}}\underset{\text{components of row 2}}{M_{2j}}M_{3k}$

Exchanging row 1 and row 2 is equivalent to exchanging $i \leftrightarrow j$ in the matrix components, i.e. $\det M \to \epsilon_{ijk}M_{1j}M_{2i}M_{3k}$

Replacing $i \leftrightarrow j$ as dummy indices, $\det M \to \epsilon_{jik}M_{1i}M_{2j}M_{3k}$

$= -\det M$ since $\epsilon_{jik} = -\epsilon_{ijk}$. Similarly for any other row exchanges.

2. b) $\det(M^T) = \frac{1}{6}\epsilon_{ijk}\epsilon_{lmp} M^T_{il} M^T_{jm} M^T_{kp}$

$= \frac{1}{6}\epsilon_{ijk}\epsilon_{lmp} M_{li} M_{mj} M_{pk}$

$= \det M$

c) $\det(M^+) = \frac{1}{6}\epsilon_{ijk}\epsilon_{lmp}\overline{(M^T{}_{il})}\,\overline{(M^T{}_{jm})}\,\overline{(M^T{}_{kp})}$

$= \frac{1}{6}\epsilon_{ijk}\epsilon_{lmp}\overline{M_{li}}\;\overline{M_{mj}}\;\overline{M_{pk}}$

$= \left(\frac{1}{6}\epsilon_{ijk}\epsilon_{lmp} M_{li} M_{mj} M_{pk}\right)^*$ since ϵ_{ijk} real.

$= (\det M)^*$

<u>Oops!</u> Forgot problem 10.5.7:

10.5.7 a) $\epsilon_{ijk}\epsilon_{pjq} = \epsilon_{jik}\epsilon_{jpq}$

$= \delta_{ip}\delta_{kq} - \delta_{iq}\delta_{kp}$

b) $\epsilon_{abc}\epsilon_{pqc} = \epsilon_{cab}\epsilon_{cpq}$

$= \delta_{ap}\delta_{bq} - \delta_{aq}\delta_{bp}$