Phys 475 S'10 Problem Set 10 Solutions

12.11.4 $x^2 y'' - 6y = 0$

Assume $y = \sum_{n=0}^{\infty} a_n x^{n+s}$, $a_0 \neq 0$

$$y' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2}$$

$$x^2 y'' = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s}$$

$$x^2 y'' - 6y = \sum_{n=0}^{\infty} \left[a_n (n+s)(n+s-1) - 6a_n \right] x^{n+s} = 0$$

$$\rightarrow a_n \left[(n+s)(n+s-1) - 6 \right] = 0$$

$a_0 \neq 0 \rightarrow s(s-1) - 6 = 0 \rightarrow s^2 - s - 6 = 0$

$$s = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} = \boxed{-2 \text{ or } 3}$$

$s = -2$: $a_1 \left[(-1)(-2) - 6 \right] = 0 \rightarrow a_1 = 0$

Similarly, $a_n = 0$ except when $(n+s)(n+s-1) - 6 = 0$, in which case a_n is arbitrary.

with $s = -2$, $(n-2)(n-3) - 6 = 0$

$\rightarrow n^2 - 5n = 0 \rightarrow n = 0$ or $n = 5$

$$\rightarrow \boxed{y = a_0 x^{-2} + a_5 x^3}$$

$s = 3$: $a_1 \left[4(3) - 6 \right] = 0 \rightarrow a_1 = 0$

Similarly, $a_n = 0$ unless $(n+3)(n+2) - 6 = 0$

$\rightarrow n^2 + 5n = 0 \rightarrow n = 0$ or $\cancel{n = -5}$ not in sum.

$$\rightarrow \boxed{y = a_0 x^3}$$

General solution: $\boxed{y = A x^{-2} + B x^3}$

12.12.1 $J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p}$

$$= \sum_{n=0}^{\infty} a_n x^{2n+p} \quad \text{where} \quad a_n \equiv \frac{(-1)^n}{2^{2n+p}\,\Gamma(n+1)\Gamma(n+1+p)}$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \frac{x^{2(n+1)+p}}{x^{2n+p}} \right|$$

$$= \lim_{n\to\infty} \left| x^2 \cdot \frac{(-1)}{4} \frac{\Gamma(n+1)\Gamma(n+1+p)}{\Gamma(n+2)\Gamma(n+2+p)} \right|$$

Use $\Gamma(p+1) = p\,\Gamma(p)$.

$$\Gamma(n+2+p) = (n+1+p)\,\Gamma(n+1+p)$$

$$\Gamma(n+2) = (n+1)\,\Gamma(n+1)$$

$$\rightarrow \rho = \lim_{n\to\infty} \left| \frac{x^2}{4} \cdot \frac{1}{(n+1+p)(n+1)} \right| = 0 \quad \forall x.$$

Hence, the infinite series for $J_p(x)$ converges $\forall x$. ■

12.12.9 $\sqrt{\frac{\pi x}{2}}\, J_{1/2}(x) = \sqrt{\frac{\pi x}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+1/2)} \left(\frac{x}{2}\right)^{2n+1/2}$

Use $\Gamma(1/2) = \sqrt{\pi} \Rightarrow \Gamma(n+1+1/2) = (n+1/2)\Gamma(n+1/2) = (n+1/2)(n-1/2)\Gamma(n-1/2)$

$$= \cdots (n+1/2)(n-1/2)(n-3/2)\cdots \tfrac{1}{2}\Gamma(1/2)$$

$$= \left(\frac{1}{2}\right)^n (2n+1)(2n-1)(2n-3)\cdots 1 \cdot \sqrt{\pi}$$

$$= \left(\frac{1}{2}\right)^n \frac{(2n+2)(2n+1)(2n)(2n-1)\cdots 1\ \sqrt{\pi}}{2(n+1)\ 2(n)\ 2(n-1)\ \cdots\ 2}$$

$$\rightarrow \Gamma(n+1+1/2) = \frac{1}{4^{(n+1)}} \frac{(2n+2)!}{(n+1)!} \sqrt{\pi}$$

and $\Gamma(n+1) = n!$

$$\rightarrow \sqrt{\frac{\pi x}{2}}\, J_{1/2}(x) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{4^{(n+1)}(n+1)!}{(2n+2)!} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{4^n} \cdot \frac{1}{2^{1/2}} x^{2n+1}$$

$$\boxed{\sqrt{\frac{\pi x}{2}}\, J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x}$$, using $\frac{2(n+1)}{(2n+2)!} = \frac{1}{(2n+1)!}$

12.16.2 $y'' + 4x^2 y = 0.$

Compare with $y'' + \frac{1-2a}{x} y' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right] y = 0,$

which has solutions $y = x^a Z_p(bx^c)$.

No y' term $\to$ $1-2a = 0 \to \boxed{a = 1/2}$

No $\frac{1}{x^2} y$ term $\to$ $a^2 - p^2c^2 = 0 \to \boxed{p^2c^2 = 1/4}$

$4x^2 = b^2c^2 x^{2c-2} \to \boxed{c = 2,\ b = 1}$

$\to \boxed{p = 1/4}$

$\to \boxed{y = x^{1/2} J_{1/4}(x^2)}$ and $\boxed{y = x^{1/2} Y_{1/4}(x^2)}$,

and superpositions.

14.1.21 $e^{iz} = e^{i(x+iy)} = e^{-y} e^{ix} = e^{-y}\cos x + i e^{-y} \sin x$
$= u + iv$, with $\boxed{u = e^{-y}\cos x}$ and $\boxed{v = e^{-y}\sin x}$

14.2.23 $\frac{x - iy}{x^2+y^2} = u + iv$ w/ $u = \frac{x}{x^2+y^2}$, $v = -\frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

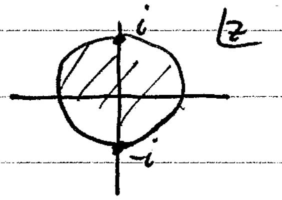
$\frac{\partial v}{\partial y} = -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = -\frac{(x^2-y^2)}{(x^2+y^2)^2} = \frac{\partial u}{\partial x}$ ✓

$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$, $\frac{\partial v}{\partial x} = +\frac{2xy}{(x^2+y^2)^2} = -\frac{\partial u}{\partial y}$ ✓

$\frac{x-iy}{x^2+y^2}$ is analytic (except @ $z=0$).

14.2.36 $\sqrt{1+z^2}$ has no derivative @ $z = \pm i$

$\rightarrow \sqrt{1+z^2}$ has singularities (branch pts.) @ $z = \pm i$



Disk of convergence of power series for $\sqrt{1+z^2}$ has radius 1.

$$\sqrt{1+z^2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$$

14.2.45 $f(z) = u + iv, \quad \vec{F} = v\,\hat{i} + u\,\hat{j}$

$$\nabla \cdot \vec{F} = 0 \iff \boxed{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0}$$

$$\nabla \times \vec{F} = 0 \iff \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v & u & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = 0$$

$$\rightarrow \boxed{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0}$$

The two boxed eqs. are the Cauchy–Riemann conditions. ∎

14.2.60 $u(x,y) = \ln(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{2}{x^2+y^2} - \frac{(2x)^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{2}{x^2+y^2} - \frac{(2y)^2}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \rightarrow u \text{ is a harmonic fn.}$$

To solve for $v(x,y)$, use Cauchy-Riemann conditions.

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \rightarrow v = 2\arctan(y/x) + f(x)$$

$$\frac{\partial v}{\partial x} = \frac{2(-y/x^2)}{1+y^2/x^2} + f'(x) = \frac{-2y}{x^2+y^2} + f'(x)$$

Cauchy-Riemann: $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{-2y}{x^2+y^2} \Rightarrow f'(x) = 0$

$f(x) = \text{const.}$

$$\boxed{v(x,y) = 2\arctan(y/x) + \text{const.}}$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{(-2y)(-2x)}{(x^2+y^2)^2} + \frac{(2x)(-2y)}{(x^2+y^2)^2} = 0$$

$\rightarrow v(x,y)$ is a harmonic function. ✓

$$f(z) = \ln(x^2+y^2) + i\,2\arctan(y/x) + \text{const.}$$
$$= \ln(z^2) + \text{const.}$$