Phys 475 Final Exam Solutions
S'10

1. a) $i^{-i} = \exp[-i \ln i]$

$= \exp[-i \ln \exp(\frac{\pi i}{2})]$

$= \exp[-i(\frac{\pi i}{2} + 2n\pi i)]$

$= \boxed{\exp[\frac{\pi}{2} + 2n\pi], \quad n = \text{integer}}$

b) $[\nabla \times (\nabla \times \vec{V})]_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \vec{V})_k$

$= \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} \frac{\partial}{\partial x_l} V_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} V_m$

$= \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_i} V_m - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} V_i$

$= \frac{\partial}{\partial x_i}\left(\frac{\partial V_m}{\partial x_m}\right) - \frac{\partial^2}{\partial x_j \partial x_j} V_i$

$= [\nabla(\nabla \cdot \vec{V})]_i - [\nabla^2 \vec{V}]_i$

$\Rightarrow \nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$ ☒

c) H Hermitian $\rightarrow H^\dagger = H$. $Ht/\hbar$ also Hermitian if $t/\hbar$ real.

$\exp(-iHt/\hbar) = \sum_{n=0}^{\infty} (-it/\hbar)^n \frac{1}{n!} H^n \equiv U$

$U^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!} (it/\hbar)^n (H^n)^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!} (it/\hbar) \underbrace{(H^\dagger H^\dagger \ldots H^\dagger)}_{n \text{ times}}$

$= \exp(iHt/\hbar)$

$U^\dagger U = \exp(iHt/\hbar) \exp(-iHt/\hbar) = \exp(0) = 1.$

$\Rightarrow$ $\boxed{U \text{ is unitary.}}$

2. a) $-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$

$\psi_1(x)$ is a sol'n w/ $E=E_1$,
$\psi_2(x)$ " w/ $E=E_2$

$$\psi_2(x)\left[-\frac{\hbar^2}{2m}\psi_1''(x) + V(x)\psi_1(x)\right] = E_1\psi_2(x)\psi_1(x)$$

$$\psi_1(x)\left[-\frac{\hbar^2}{2m}\psi_2''(x) + V(x)\psi_2(x)\right] = E_2\psi_1(x)\psi_2(x)$$

Taking the difference of these 2 eqs,

$$-\frac{\hbar^2}{2m}\left[\psi_2(x)\psi_1''(x) - \psi_1(x)\psi_2''(x)\right] = (E_1-E_2)\psi_1(x)\psi_2(x)$$

$$-\frac{\hbar^2}{2m}\frac{d}{dx}\left(\psi_2(x)\psi_1'(x) - \psi_1(x)\psi_2'(x)\right) = (E_1-E_2)\psi_1(x)\psi_2(x)$$

Integrate from a to b:

$$-\frac{\hbar^2}{2m}\left(\psi_2(x)\psi_1'(x) - \psi_1(x)\psi_2'(x)\right)\Big|_a^b = (E_1-E_2)\int_a^b \psi_1(x)\psi_2(x)dx$$

$\Rightarrow$ If $E_1 \neq E_2$ and $\left(\psi_2(x)\psi_1'(x) - \psi_1(x)\psi_2'(x)\right)\Big|_a^b = 0$

then $\int_a^b \psi_1(x)\psi_2(x)\,dx = 0$

b) $f(x) = \sum_n a_n \psi_n(x)$

Multiply by $\psi_m(x)$ and integrate from a to b:

$$\int_a^b f(x)\,\psi_m(x)\,dx = \sum_n a_n \int_a^b \psi_m(x)\,\psi_n(x)\,dx$$

$$= \sum_n a_n \delta_{mn} = a_m$$

$$\Rightarrow \boxed{a_n = \int_a^b f(x)\,\psi_n(x)\,dx}$$

c) $$\int_a^b |f(x)|^2 dx = \int_a^b \left(\sum_n a_n \psi_n(x)\right)\left(\sum_m a_m^* \psi_m(x)\right) dx$$

$$= \int_a^b \sum_{n,m} a_n a_m^* \psi_n(x)\,\psi_m(x)\,dx$$

$$= \sum_{n,m} a_n a_m^* \delta_{mn}$$

$$= \sum_n a_n a_n^* = \sum_n |a_n|^2$$

$$\Rightarrow \int_a^b |f(x)|^2 dx = \sum_n |a_n|^2$$

3. a) $u(x,y) = xy$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{Cauchy - Riemann Conditions}$$

$$\frac{\partial u}{\partial x} = y \rightarrow \frac{\partial v}{\partial y} = y, \quad v = \frac{y^2}{2} + f(x)$$

$$\frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial x} = f'(x) \rightarrow \quad f'(x) = -x$$

$$f(x) = -\frac{x^2}{2} + \text{Const.}$$

$$\boxed{v(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \text{const.}}$$

$$f(z) = xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right) + \text{const.}$$

$$z = (x+iy) \rightarrow z^2 = x^2 - y^2 + 2ixy$$

$$-i\frac{z^2}{2} = xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right)$$

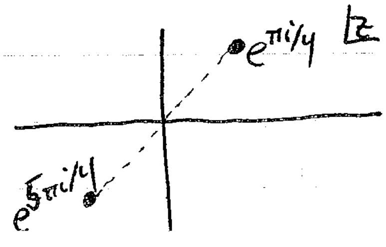
$$\rightarrow \boxed{f(z) = -i\frac{z^2}{2} + \text{const.}}$$

b) $f(z) = \dfrac{e^{iaz}}{z^2 - i}$

Singularities are at $z^2 - i = 0 \rightarrow z^2 = e^{\frac{\pi i}{2}}$

$$z = e^{\pi i/4 + n\pi i}$$

$$z = e^{\pi i/4} \quad \text{or} \quad z = e^{5\pi i/4}$$



3. c) $f(z) = \dfrac{e^{iaz}}{(z-e^{\pi i/4})(z-e^{5\pi i/4})}$

$$\text{Res}\left(e^{\pi i/4}\right) = \lim_{z\to e^{\pi i/4}} \left(z-e^{\pi i/4}\right) f(z)$$

$$= \frac{e^{iae^{\pi i/4}}}{e^{\pi i/4}-e^{5\pi i/4}} = \frac{e^{ia\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}}{e^{\pi i/4}\left(1-e^{\pi i}\right)}$$

$$= \boxed{\frac{e^{-\frac{\sqrt{2}}{2}a}\, e^{i\left(a\frac{\sqrt{2}}{2}-\pi/4\right)}}{2}}$$

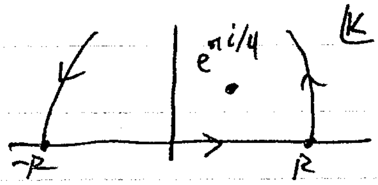
$$\text{Res}\left(e^{5\pi i/4}\right) = \lim_{z\to e^{5\pi i/4}} \left(z-e^{5\pi i/4}\right) f(z)$$

$$= \frac{e^{ia\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}}{e^{5\pi i/4}-e^{\pi i/4}} = \frac{e^{a\sqrt{2}/2}\, e^{-ia\sqrt{2}/2}}{e^{\pi i/4}\left(e^{\pi i}-1\right)}$$

$$= \boxed{-\frac{e^{\frac{\sqrt{2}}{2}a}\, e^{-i\left(a\frac{\sqrt{2}}{2}+\pi/4\right)}}{2}}$$

d) $g(k) = \dfrac{1}{k^2-i}$, $f(x) = \displaystyle\int_{-\infty}^{\infty} dk\, e^{ikx} \frac{1}{k^2-i}$

$x>0$: Close contour in upper half plane



$$f(x) = \lim_{R\to\infty} \oint dk\, \frac{e^{ikx}}{k^2-i} = 2\pi i \cdot \text{Res}\left(k = \pi i/4\right)$$

$$= \boxed{\frac{2\pi i\, e^{-\frac{\sqrt{2}}{2}x}\, e^{i\left(x\frac{\sqrt{2}}{2}-\pi/4\right)}}{2}}$$

4. a) $xy'' + (1-x)y' + py = 0$

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}$$

$$x y''(x) = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-1}$$

$$(1-x) y'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n n x^n$$

$$xy'' + (1-x)y' + py = \sum_{n=0}^{\infty} \Big[a_n n(n-1) x^{n-1} + a_n n x^{n-1} - a_n n x^n + a_n p x^n \Big] = 0$$

The coefficient of each power of x must vanish:

Coefficient of:	x^0	x^1	x^2	...	x^n
	$a_1 + p a_0$	$4a_2 + (p-1)a_1$	$9a_3 + (p-2)a_2$		$(1+n)^2 a_{n+1} + (p-n)a_n$

$$\boxed{a_{n+1} = -\frac{(p-n)\,a_n}{(1+n)^2}}$$ Recursion relation.

b) If p is an integer, then a_{p+1} vanishes, as does every a_n w/ $n > p$.

Hence, $y(x)$ is a polynomial of order p.