The Invariant Interval
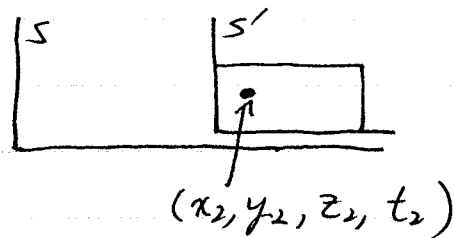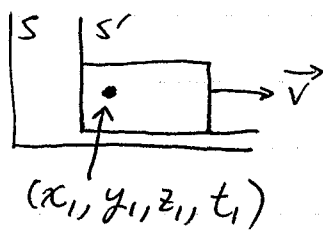
Consider two events that occur at the same position in some inertial frame S', with coordinates (x', y', z') at times t_1' and t_2'.

The proper time between the two events is $\boxed{\Delta t' = t_2' - t_1'}$.

Since the locations of the two events are the same, $\boxed{\Delta x' = \Delta y' = \Delta z' = 0}$.

Now consider the coordinates in a frame S with respect to which the frame S' is moving to the right along the x-axis with velocity $\vec{v}$, in the standard config.

(x_1, y_1, z_1, t_1) (x_2, y_2, z_2, t_2)

By the Lorentz transformations:

$$x_1 = \gamma(x_1' + vt_1')$$
$$y_1 = y_1'$$
$$z_1 = z_1'$$
$$t_1 = \gamma\left(t_1' + \frac{vx_1'}{c^2}\right)$$

$$x_2 = \gamma(x_2' + vt_2')$$
$$y_2 = y_2'$$
$$z_2 = z_2'$$
$$t_2 = \gamma\left(t_2' + \frac{vx_2'}{c^2}\right)$$

where $x_1' = x_2' = x'$, $y_1' = y_2' = y'$, $z_1' = z_2' = z'$.

The time between the two events as measured by observers in the frame S is:

$$\boxed{\Delta t} = t_2 - t_1 = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \boxed{= \gamma \Delta t'} \quad \left(\begin{array}{c}\text{Time} \\ \text{Dilation}\end{array}\right)$$

The distance between the events in the frame S is:

$$\boxed{\Delta x} = x_2 - x_1 = \gamma(\Delta x' + v\Delta t') \boxed{= \gamma v \Delta t'}.$$

Note that this is also $\Delta x = v\Delta t$, which is what you would expect since S' moves w/ speed v w/ respect to S.

Also, $\boxed{\Delta y = y_2 - y_1 = 0}$ and $\boxed{\Delta z = z_2 - z_1 = 0}$

We define the <u>invariant interval</u> $(\Delta s)^2$ as

$$\boxed{(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2}$$

$$= \gamma^2 c^2 (\Delta t')^2 - \gamma^2 v^2 (\Delta t')^2$$
$$= \gamma^2 c^2 \left(1 - \frac{v^2}{c^2}\right)(\Delta t')^2$$
$$= \frac{1}{(1 - v^2/c^2)} c^2 \left(1 - \frac{v^2}{c^2}\right)(\Delta t')^2$$

$$\boxed{= c^2 (\Delta t')^2}$$

In the frame S', we could define
$$\boxed{(\Delta s')^2} = c^2(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$$
$$\boxed{= c^2 (\Delta t')^2}$$

It should be clear why $(\Delta s)^2$ is called the
__invariant__ interval — It is the same in all
frames.

Note that $(\Delta s)^2$ is also invariant under rotations,
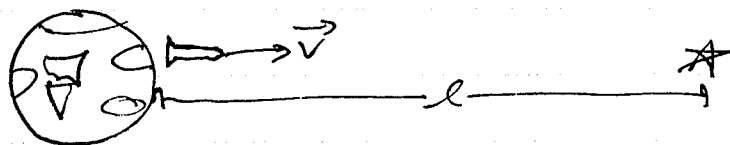i.e. it is a __scalar__ under rotations:

$\quad (\Delta t)^2$ is invariant

$(\Delta \vec{r})^2 \equiv (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ is also invariant.

Hence, $(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta \vec{r})^2$ is invariant.

Physically, $(\Delta s)^2 = c^2 (\text{proper time between events})^2$.
Hence, $(\Delta s)^2$ determines the time elapsed on the
watch of the ∧ inertial observer for whom the two events
occur at the same place, if such an observer exists.

Example: The twin paradox again.

In Earth's frame, $\Delta t (\text{Earth} \to \text{Star}) = \dfrac{\ell}{v}$

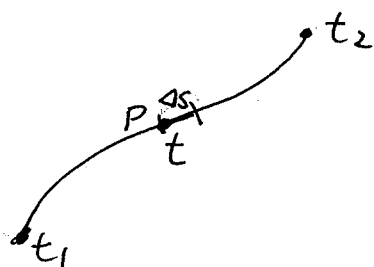$\qquad\qquad\qquad\qquad \Delta x (\text{Earth} \to \text{Star}) = \ell$

The proper time elapsed on the rocket's clock is

$\tau = \dfrac{1}{c}\sqrt{(\Delta s)^2} = \dfrac{1}{c}\sqrt{c^2(\Delta t)^2 - (\Delta x)^2} = \dfrac{1}{c}\sqrt{\dfrac{\ell^2 c^2}{v^2} - \ell^2}$

$\boxed{\tau} = \dfrac{\ell}{v}\sqrt{1 - v^2/c^2} = \dfrac{\ell}{v\gamma} \boxed{= \dfrac{\Delta t}{\gamma}}$

Locally Inertial Frames

Suppose we would like to know the time elapsed on the watch of an observer moving along some trajectory $\vec{r}(t)$, not necessarily inertial.

Near each point P along the trajectory there is an inertial reference frame moving w/ the velocity along the trajectory at that point.

To calculate the proper time elapsed over an infinitessimal portion of the trajectory we can use the invariant interval:

$$\Delta \tau = \frac{1}{c} \Delta s = \frac{1}{c} \sqrt{c^2 (\Delta t)^2 - (\Delta \vec{r})^2}$$

$$= \Delta t \sqrt{1 - \frac{1}{c^2} \left(\frac{\Delta \vec{r}}{\Delta t}\right)^2}$$

$$\boxed{\Delta \tau = \Delta t \sqrt{1 - v(t)^2/c^2}}$$

where $v(t)$ is the speed along the trajectory at time t. For infinitessimal quantities it is natural to write

$$\boxed{d\tau = dt \sqrt{1 - v(t)^2/c^2}}$$

The total time elapsed on the clock moving along the trajectory is called the proper time associated with that trajectory, and is given by

$$\boxed{\tau = \int_{t_1}^{t_2} d\tau = \int_{t_1}^{t_2} dt \sqrt{1 - v(t)^2/c^2}}$$

Example: Circular motion

What is the time elapsed on the watch of an observer moving in a circle of radius r with angular rate ω (as measured in some inertial frame), in one period of the motion?

Trajectory described by
$$x = r \cos(\omega t)$$
$$y = r \sin(\omega t)$$

$$\frac{dx}{dt} = -r\omega \sin(\omega t)$$

$$\frac{dy}{dt} = r\omega \cos(\omega t)$$

$$\left(\frac{d\vec{r}}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2\omega^2$$

Proper time $\tau = \int_0^T dt \sqrt{1 - v(t)^2/c^2}$

$$= \int_0^T dt \sqrt{1 - \left(\frac{r\omega}{c}\right)^2}$$

$$= T \sqrt{1 - \left(\frac{r\omega}{c}\right)^2}$$

The period $T = \frac{2\pi}{\omega}$. Hence, in one period the time elapsed on the watch moving along the circular trajectory is

$$\boxed{\tau = \frac{2\pi}{\omega} \sqrt{1 - \left(\frac{r\omega}{c}\right)^2}}$$

Causality and the Light Cone

The ordering of events is not necessarily the same in different reference frames. Does that mean that in some frame someone could win the lottery and later pick the winning ticket? We are used to thinking of certain events as causing other events. It would be quite confusing if the notion of causality had to be modified due to special relativity.

Consider two events (x_1, y_1, z_1) at time t_1
and (x_2, y_2, z_2) at time t_2.

In another frame in the standard configuration,
$$t_1' = \gamma\left(t_1 - \frac{v x_1}{c^2}\right) , \qquad t_2' = \gamma\left(t_2 - \frac{v x_2}{c^2}\right)$$

The difference in time between the two events in the first frame is $\Delta t = t_2 - t_1$. If $\Delta t > 0$ then the second event occurs after the first in that frame.

In the other frame, $\Delta t' = t_2' - t_1'$
$$= \gamma\left(\Delta t - \frac{v}{c^2} \Delta x\right)$$
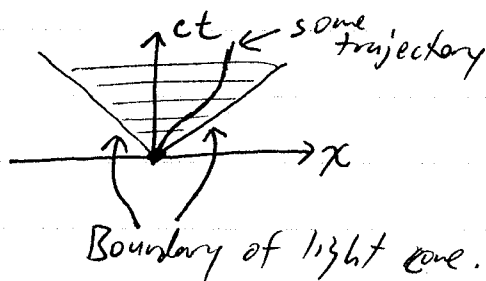$$= \gamma \Delta t \left(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}\right)$$

The sign of $\Delta t'$ is the same as the sign of Δt as long as
$$\left(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}\right) > 0.$$

Since $v \leq c$, we learn that the only way the ordering of events can be reversed by a Lorentz boost is if $\frac{\Delta x}{\Delta t} > c$, i.e. the average speed of an observer moving between the two events would have to be bigger than c.
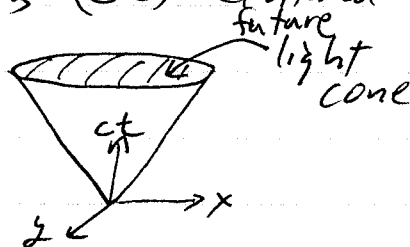
Causality is saved in special relativity because nothing can travel faster than light.

To describe all possible future events that can be affected by an event at time t at a certain location, we can construct the future light cone.
Consider the projection of some trajectory onto the x-axis as a function of time. The trajectory must lie inside the future light cone bounded by the line $x = \pm ct$:
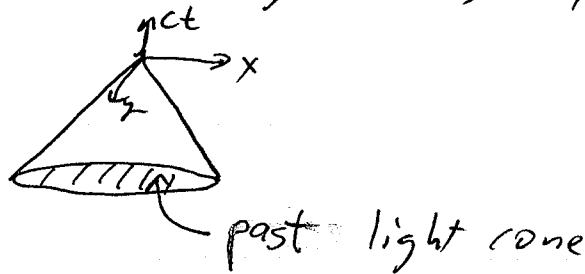

Boundary of light cone.

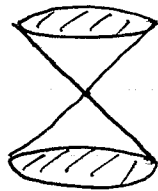At each future time t, the light cone is really a sphere of radius (ct) centered on the initial event.

All events in the past that could have affected
an event at a given time and place lie in the
<u>past light cone</u>. The past light cone is constructed
just like the future light cone, only in reverse:

past light cone

The combination of the past and future light
cones of some event are just called the light cone:

In any frame, all events in the future light cone
of an event occur <u>after</u> that event. Similarly,
all events in the past light cone of an event occur
<u>before</u> that event.

The boundary of the light cone is the surface
$$c^2(\Delta t)^2 - (\Delta \vec{r})^2 = 0.$$
Since this is just the invariant interval, it is the
same in all frames. Hence, <u>the boundary of the</u>
<u>light cone is Lorentz invariant.</u>