Phys 201 - Problem Set 9 Solutions
F'09

6.36 a) $F(x) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n\pi x}{\lambda}\right)$

Multiply both sides by $\sin\left(\frac{2m\pi x}{\lambda}\right)$ and integrate

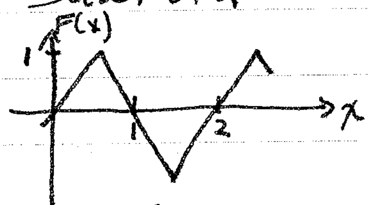
$$\int_0^\lambda dx\, F(x) \sin\left(\frac{2m\pi x}{\lambda}\right) = \int_0^\lambda dx \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n\pi x}{\lambda}\right) \sin\left(\frac{2m\pi x}{\lambda}\right)$$

$$= \sum_{n=0}^{\infty} B_n \underbrace{\int_0^\lambda dx \sin\left(\frac{2n\pi x}{\lambda}\right) \sin\left(\frac{2m\pi x}{\lambda}\right)}_{\substack{= \frac{\lambda}{2} \text{ if } n=m \\ 0 \text{ if } n\neq m}}$$

$$= B_m \frac{\lambda}{2}$$

$$\Rightarrow \quad B_m = \frac{2}{\lambda} \int_0^\lambda dx\, F(x) \sin\left(\frac{2m\pi x}{\lambda}\right)$$

b) Sawtooth



period: $\lambda = 2$

$$F(x) = \begin{cases} 2x, & 0 < x < \frac{1}{2} \\ -2x+2, & \frac{1}{2} < x < \frac{3}{2} \\ 2x-4, & \frac{3}{2} < x < 2 \end{cases}$$

$$B_m = \frac{2}{2}\int_0^{1/2} dx\, 2x \sin\left(\frac{2m\pi x}{2}\right) + \frac{2}{2}\int_{1/2}^{3/2} dx\, (-2x+2) \sin\left(\frac{2m\pi x}{2}\right)$$

$$+ \frac{2}{2}\int_{3/2}^{2} dx\, (2x-4) \sin\left(\frac{2m\pi x}{2}\right)$$

To evaluate $\int dx\, x \sin\left(\frac{2m\pi x}{2}\right)$ integrate by parts:

$$\int dx\, x \sin(m\pi x) = -\frac{1}{m\pi} x \cos m\pi x + \int dx \frac{1}{m\pi} \cos(m\pi x)$$

$$= \frac{-1}{m\pi} x \cos(m\pi x) + \frac{1}{(m\pi)^2} \sin(m\pi x)$$

$$B_m = 2\left[-\frac{1}{m\pi} x \cos(m\pi x) + \frac{1}{(m\pi)^2} \sin(m\pi x)\right]\Bigg|_0^{1/2}$$

$$+ 2\left[+\frac{1}{m\pi} x \cos(m\pi x) - \frac{1}{(m\pi)^2} \sin(m\pi x) + \frac{1}{m\pi}(-\cos(m\pi x))\right]\Bigg|_{1/2}^{3/2}$$

$$+2\left[-\frac{1}{m\pi} x \cos(m\pi x) + \frac{1}{(m\pi)^2} \sin(m\pi x) - \frac{2}{m\pi}(-\cos(m\pi x))\right]\Bigg|_{3/2}^{2}$$

$$= 2\left[\frac{-1}{2m\pi} \cos\left(\frac{m\pi}{2}\right) + \frac{1}{(m\pi)^2} \sin\left(\frac{m\pi}{2}\right)\right]$$

$$+ 2\left[\frac{1}{m\pi} \cdot \frac{3}{2} \cos\left(\frac{3m\pi}{2}\right) - \frac{1}{(m\pi)^2} \sin\left(\frac{3m\pi}{2}\right) - \frac{1}{m\pi} \cos\left(\frac{3m\pi}{2}\right)\right]$$

$$- 2\left[\frac{1}{m\pi} \cdot \frac{1}{2} \cos\left(\frac{m\pi}{2}\right) - \frac{1}{(m\pi)^2} \sin\left(\frac{m\pi}{2}\right) - \frac{1}{m\pi} \cos\left(\frac{m\pi}{2}\right)\right]$$

$$+2\left[-\frac{1}{m\pi} \cdot 2 \cos 2m\pi + \frac{1}{(m\pi)^2} \sin 2m\pi + \frac{2}{m\pi} \cos 2m\pi\right]$$

$$-2\left[-\frac{1}{m\pi} \cdot \frac{3}{2} \cos \frac{3m\pi}{2} + \frac{1}{(m\pi)^2} \sin \frac{3m\pi}{2} + \frac{2}{m\pi} \cos \frac{3m\pi}{2}\right]$$

Use $\cos \frac{3m\pi}{2} = \cos\left(2m\pi - \frac{m\pi}{2}\right) = \overbrace{\cos 2m\pi}^{1} \cos \frac{m\pi}{2} + \overbrace{\sin 2m\pi}^{0} \sin \frac{m\pi}{2}$

$$= \cos \frac{m\pi}{2}$$

$$\sin \frac{3m\pi}{2} = \sin\left(2m\pi - \frac{m\pi}{2}\right) = -\sin \frac{m\pi}{2}$$

$$B_m = 2\cos\left(\frac{m\pi}{2}\right)\underbrace{\left[-\frac{1}{2m\pi}+\frac{3}{2m\pi}-\frac{1}{m\pi}-\frac{1}{2m\pi}+\frac{1}{m\pi}+\frac{3}{2m\pi}-\frac{2}{m\pi}\right]}_{0}$$

$$+2\left[\cancel{-\frac{2}{m\pi}+\frac{2}{m\pi}}\right]$$

$$+2\sin\left(\frac{m\pi}{2}\right)\left[\frac{1}{(m\pi)^2}+\frac{1}{(m\pi)^2}+\frac{1}{(m\pi)^2}+\frac{1}{(m\pi)^2}\right]$$

$$= \frac{8}{m^2\pi^2}\begin{cases} 0 & \text{if } m \text{ even} \\ (-1)^{\frac{m-1}{2}} & \text{if } m \text{ odd}\end{cases}$$

6.37 $\Delta x = 6\times 10^{-15}\ m$

$$\Delta p \gtrsim \frac{\hbar}{2\Delta x} \Rightarrow \Delta v \gtrsim \frac{\hbar}{2m\Delta x}$$

$$= \frac{6.58\times 10^{-16}\ eV\cdot s}{2\cdot(938\times 10^{6}\ eV/c^2)(6\times 10^{-15}\ m)}$$

$$= \frac{6.58}{(12)(938)}\,10^{-7}\ \frac{s}{m}\cdot(3\times 10^{8}\ m/s)\cdot c$$

$$= 0.018\ c$$

$$\Rightarrow \boxed{\Delta v \gtrsim 0.018\ c}$$

6.44 $E \sim (\Delta p)c \sim \frac{\hbar c}{2\Delta x} = m_{eff} c^2$

$m_{eff} \sim \frac{\hbar}{2c\Delta x}$

a) $\frac{1}{2} m v_{esc}^2 = \frac{GMm}{R}$ conservation of energy

$\rightarrow \boxed{v_{esc} = \sqrt{2GM/R}}$

b) $M \rightarrow m_{eff} \sim \frac{\hbar}{2c\Delta x} \rightarrow \frac{\hbar}{2cL_{Pl}}$

$R \rightarrow \Delta x \rightarrow L_{Pl}$

$v_{esc} \rightarrow c$

$c = \sqrt{\frac{2G \cdot \hbar}{2c L_{Pl}^2}} \Rightarrow \boxed{L_{Pl} = \sqrt{\frac{G\hbar}{c^3}}}$

c) $L_{Pl} = \sqrt{\frac{(6.67 \times 10^{-11}\ \text{J m/kg}^2)(1.05 \times 10^{-34}\ \text{J s})}{(3 \times 10^8\ \text{m/s})^3}}$

$= \boxed{1.6 \times 10^{-35}\ \text{m}}$

d) $2\ \text{fm} = 2 \times 10^{-15}\ \text{m} \simeq \boxed{1.25 \times 10^{20}\ L_{Pl}}$

7.17 $a = 0.2\ \text{nm}, \quad E_n = \dfrac{\hbar^2 n^2 \pi^2}{2ma^2}$

$$E_1 = \frac{\hbar^2 \pi^2}{2ma^2} = \frac{(6.58\times10^{-16}\ \text{eV}\cdot\text{s})^2\pi^2}{2\left(0.511\times10^6\ \text{eV}/(3\times10^8\ \text{m/s})^2\right)(0.2\times10^{-9}\ \text{m})^2}$$

$$= \boxed{9.4\ \text{eV}}$$

$$E_2 = 4E_1 = \boxed{37.6\ \text{eV}}$$

$$E_3 = 9E_1 = \boxed{84.6\ \text{eV}}$$

7.24 $\psi = Ae^{ikx} + Be^{-ikx}$

$$\psi' = A(ik)e^{ikx} + B(-ik)e^{-ikx}$$

$$\psi'' = A(ik)^2e^{ikx} + B(-ik)^2e^{-ikx}$$

$$= -k^2\left(Ae^{ikx} + Be^{-ikx}\right)$$

$$= -k^2\psi$$