Phys 201 F'09 Problem Set 11 Solutions

1. a) Expand $U(x)$ about the minimum $x = x_0$.

$$U(x) = U(x_0) + \underbrace{U'(x_0)}_{\text{0 at local minimum}} (x - x_0) + \frac{1}{2} U''(x_0)(x-x_0)^2 + \underbrace{\mathcal{O}(x-x_0)^3}_{\text{small, if } |x-x_0| \text{ is small enough}}$$

$$\Rightarrow \boxed{U(x) \approx U(x_0) + \frac{1}{2} U''(x_0)(x-x_0)^2} \quad \text{for small } |x - x_0|$$

This is of the form $U(x) = \text{const.} + \frac{1}{2} K (x-x_0)^2$, with

const. $= U(x_0)$

$$\boxed{K = U''(x_0)}$$

b) $\psi_0(x) = A \exp\left(-x^2/2b^2\right)$, $b = \sqrt{\frac{\hbar}{m\omega_c}}$, $\omega_c = \sqrt{\frac{K}{m}}$.

$$\psi_0'(x) = -A \frac{x}{b^2} \exp\left(-x^2/2b^2\right)$$

$$\psi_0''(x) = -\frac{A}{b^2} \exp\left(-\frac{x^2}{2b^2}\right) + A \frac{x^2}{b^4} \exp\left(-\frac{x}{2b^2}\right)$$

$$= \left(-\frac{1}{b^2} + \frac{x^2}{b^4}\right) \psi_0(x)$$

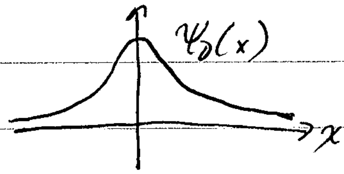
$$= \left(-\frac{m\omega_c}{\hbar} + \frac{m^2\omega_c^2}{\hbar^2} x^2\right) \psi_0(x)$$

$$= \left(-\frac{m\omega_c}{\hbar} + \frac{mK}{\hbar^2} x^2\right) \psi_0(x)$$

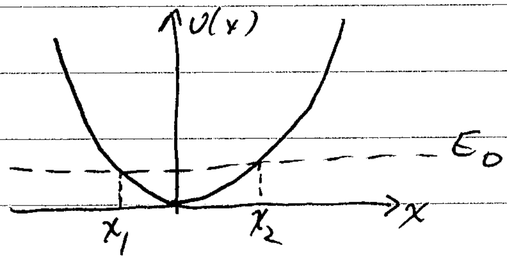
$$-\frac{\hbar^2}{2m} \psi_0''(x) + \frac{1}{2} K x^2 \psi_0(x) = -\frac{\hbar^2}{2m}\left(-\frac{m\omega_c}{\hbar} + \frac{mK}{\hbar^2} x^2\right)\psi_0(x) + \frac{1}{2} K x^2 \psi_0(x)$$

$$= \frac{\hbar\omega_c}{2} \psi_0(x) \equiv E_0 \psi_0(x)$$

Hence, $\psi_0(x)$ is the wavefunction of an energy eigenstate with energy $\boxed{E_0 = \frac{\hbar\omega_c}{2}}$

c) 

Since $\psi_0(x)$ has no nodes, it must describe the ground state.

d) 

The turning points x_1 and x_2 are the values of x such that $U(x) = E_0$

$$\frac{1}{2}Kx^2 = E_0 \rightarrow \boxed{x_1 = -\sqrt{\frac{2E_0}{K}}, \quad x_2 = +\sqrt{\frac{2E_0}{K}}}$$

or $\boxed{x_1 = -\sqrt{\frac{\hbar\omega_c}{K}}, \quad x_2 = +\sqrt{\frac{\hbar\omega_c}{K}}}$

e) $\mathrm{Prob}(x < x_1 \text{ or } x > x_2)$

$$= \int_{-\infty}^{x_1} dx\, |\psi_0(x)|^2 + \int_{x_2}^{\infty} dx\, |\psi_0(x)|^2$$

$$= \boxed{\int_{-\infty}^{x_1} dx\, |A|^2 e^{-x^2/b^2} + \int_{x_2}^{\infty} dx\, |A|^2 e^{-x^2/b^2}}$$

Since A must be normalized so that $\int_{-\infty}^{\infty} dx\, |\psi_0(x)|^2 = 1$, we can also write this as

$$\mathrm{Prob}(x < x_1 \text{ or } x > x_2) = 1 - \mathrm{Prob}(x_1 < x < x_2)$$

$$= \boxed{1 - \int_{x_1}^{x_2} dx\, |A|^2 e^{-x^2/b^2}}$$