Physics 201, Fall 2009
Problem Set 11
Due Friday, November 20.

The Simple Harmonic Oscillator

The simple harmonic oscillator appears in every branch of physics. Classical oscillations of a system about an equilibrium configuration are generically described by a harmonic oscillator potential as long as the oscillations are small enough. Here you will compare some features of the classical and quantum mechanical harmonic oscillator.

a) Consider some potential energy function $U(x)$ with local minimum at x_0, in other words $U'(x_0) = 0$ and $U''(x_0) > 0$. By considering the Taylor expansion of $U(x)$ about x_0, show that small oscillations about the minimum are described by a potential energy function of the form,

$$U(x) \approx (\text{const}) + \frac{1}{2}k(x - x_0)^2.$$

What is the "force constant" k in terms of derivatives of $U(x)$?

b) Consider the time-independent Schrödinger equation for a particle with mass m and harmonic oscillator potential $U(x) = \frac{1}{2}kx^2$.

Define $b = \sqrt{\hbar/(m\omega_c)}$, where $\omega_c = \sqrt{k/m}$. Show that $\psi_0(x) = Ae^{-x^2/2b^2}$ solves the one-dimensional Schrödinger equation for some energy E_0. What is the ground state energy E_0?

Notice that the quantum mechanical ground state energy is higher than the classical ground state energy. You can think of this as due to the uncertainty principle: if a particle is localized near the bottom of a potential well, then there is an uncertainty in its momentum, and hence its kinetic energy does not vanish.

c)Using the properties of the normalizable solutions to the Schrödinger equation, explain why $\psi_0(x)$ must describe the ground state.

One can show (but you don't have to) that the allowed energies are all of the simple form $E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c$.

d) Consider a classical simple harmonic oscillator with energy E_0. Classically the particle moves between two classical turning points, x_1 and x_2 with $x_1 < x_2$. What are x_1 and x_2 in terms of k and m?

e) Write an integral expression for the probability that the quantum mechanical harmonic oscillator in its ground state will be found outside the classically allowed region, *i.e.* $x < x_1$, or $x > x_2$. You do not need to evaluate the integral.