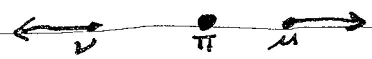
Phys 201 F'09 Final Exam Solutions

1. a)



Conservation of momentum $\vec{p}_\mu + \vec{p}_\nu = 0 \Rightarrow p_\nu = m_\mu u_\mu \gamma_\mu$

Conservation of energy $m_\mu c^2 \gamma_\mu + p_\nu c = m_\pi c^2$

$$= m_\mu c^2 \gamma_\mu + m_\mu u_\mu \gamma_\mu c$$

where $\gamma_\mu = \left(1 - \frac{u_\mu^2}{c^2}\right)^{-1/2}$

$$m_\mu c^2 \gamma_\mu \left(1 + \frac{u_\mu}{c}\right) = m_\pi c^2$$

$$m_\mu c^2 \left(\left(1 - \frac{u_\mu}{c}\right)\left(1 + \frac{u_\mu}{c}\right)\right)^{-1/2} \left(1 + \frac{u_\mu}{c}\right) = m_\pi c^2$$

$$\left(\left(1 + \frac{u_\mu}{c}\right) \Big/ \left(1 - \frac{u_\mu}{c}\right)\right)^{1/2} = \frac{m_\pi}{m_\mu}$$

$$1 + \frac{u_\mu}{c} = \frac{m_\pi^2}{m_\mu^2}\left(1 - \frac{u_\mu}{c}\right)$$

$$\frac{u_\mu}{c}\left(1 + m_\pi^2 / m_\mu^2\right) = m_\pi^2 / m_\mu^2 - 1$$

$$\boxed{\frac{u_\mu}{c} = \frac{(m_\pi / m_\mu)^2 - 1}{(m_\pi / m_\mu)^2 + 1}}$$

b) With $m_\pi = 140\ \mathrm{MeV}/c^2$, $m_\mu = 106\ \mathrm{MeV}/c^2$:

$$\frac{u_\mu}{c} = 0.27$$

c) $\gamma_\mu = \left(1 - {u_\mu^2}/{c^2}\right)^{-1/2} = 1.04$

$\tau_{rest} = 2 \times 10^{-6}$ s

$\tau_{lab} = \tau_{rest}\gamma_\mu = (2 \times 10^{-6} \text{ s})(1.04)$

$= 2.08 \times 10^{-6}$ s

2. a) $R(r) = r^{n-1} e^{-r/na_B}$, $l = n-1$

$$-\frac{\hbar^2}{2Mr}\frac{d^2}{dr^2}(rR) - \frac{ke^2}{r}R + \frac{\hbar^2(n-1)n}{2Mr^2}R$$

$$= -\frac{\hbar^2}{2Mr}\frac{d}{dr}\left(n r^{n-1} e^{-r/na_B} - \frac{r^n}{na_B}e^{-r/na_B}\right) - \frac{ke^2}{r}R + \frac{\hbar^2(n-1)n}{2Mr^2}R$$

$$= -\frac{\hbar^2}{2Mr}\left(n(n-1)r^{n-2}e^{-r/na_B} - \frac{2}{a_B}r^{n-1}e^{-r/na_B} + \frac{r^n}{n^2a_B^2}e^{-r/na_B}\right)$$
$$- \frac{ke^2}{r}r^{n-1}e^{-r/na_B} + \frac{\hbar^2(n-1)n}{2Mr^2}r^{n-1}e^{-r/na_B}$$

$$= -\frac{\hbar^2}{2M}\left(\cancel{n(n-1)r^{n-3}} - \cancel{\frac{2kMe^2}{\hbar^2}r^{n-2}} + \frac{r^{n-1}}{n^2}\frac{k^2e^4M^2}{\hbar^4}\right)e^{-r/na_B}$$
$$\cancel{-ke^2 r^{n-2} e^{-r/na_B}} + \cancel{\frac{\hbar^2(n-1)n}{2M}} r^{n-3} e^{-r/na_B}$$

$$= -\frac{Mk^2e^4}{n^2\hbar^2} r^{n-1} e^{-r/na_B}$$

$$= -E_R/n^2 \; R(r) \equiv E_n R(r)$$

b) From part a, $E_n = -E_R/n^2$

c) $m = -\ell, -(\ell-1), \ldots, \ell-1, \ell$
$ = -(n-1), -(n-2), \ldots, n-2, n-1$

d) The degeneracy of states w/ principal quantum # n is $\boxed{2n^2}$

<u>Proof</u>: For a given ℓ, $m = \underbrace{-\ell, \ldots, \ell}_{2\ell+1 \text{ possible values of } m}$

For a given n, $\ell = 0, 1, \ldots, n-1$.
possible ℓ or m is $\sum_{\ell=0}^{n-1} (2\ell+1)$.

Using $\sum_{\ell=0}^{n-1} \ell = \frac{(n-1)n}{2}$ and $\sum_{\ell=0}^{n-1} 1 = n$,

$$\sum_{\ell=0}^{n-1} (2\ell+1) = 2 \cdot \frac{(n-1)n}{2} + n = n^2$$

Since there are two possible spins $m_s = \pm \frac{1}{2}$ for each orbital, the degeneracy is $2n^2$.

e) $P(r) = 4\pi r^{2n} e^{-2r/na_B}$

$$\frac{dP}{dr} = 4\pi \cdot \left(2n r^{2n-1} - \frac{2}{na_B} r^{2n}\right) e^{-2r/na_B} = 0 \text{ @ } r_{\substack{max \\ prob.}}$$

$$\Rightarrow r_{\substack{max \\ prob}} = \underbrace{0}_{P(0)=0} \text{ or } \underbrace{n^2 a_B}_{P(n^2 a_B) > 0}$$

$$\Rightarrow \boxed{r_{\substack{max \\ prob}} = n^2 a_B}$$

This agrees w/ the radius of the n^{th} Bohr orbit.

3. a) $K_{max} = hf - \phi$

hf = energy of photon with frequency f

ϕ = work function of metal

= minimum energy required to remove an electron.

b) The alkali metals have low ionization energies because they are the first elements in each shell, and hence have the smallest (attractive) nuclear charge of each shell. The low ionization is related to the ease of removing an electron from the metal, hence the low work function.

c) 1) Increasing the intensity of light does not lead to increased energy of the ejected electrons.

2) Even at high intensity, if the frequency of light is low enough then no electrons are emitted.

d) $K_{max} = hf - \phi = \frac{hc}{\lambda} - \phi$

$$= \frac{1240 \text{ eV nm}}{200 \text{ nm}} - 4.7 \text{ eV}$$

$$= 1.5 \text{ eV}$$

To stop an electron with kinetic energy K_{max} requires an electric potential $V_s = \frac{K_{max}}{e} = \boxed{1.5 \text{ V}}$