Problems from Taylor, Zafiratos and Dubson:

6.36, 6.37

Additional Problem: *The Simple Harmonic Oscillator*

The simple harmonic oscillator appears in every branch of physics. Generically, classical oscillations of a system about an equilibrium configuration are described by a harmonic oscillator potential as long as the oscillations are small enough. Here you will compare some features of the classical and quantum mechanical harmonic oscillator.

a) Consider some potential $V(x)$ with local minimum at x_0, in other words $V'(x_0) = 0$ and $V''(x_0) > 0$. By considering the Taylor expansion of $V(x)$ about x_0, show that small oscillations are described by a potential of the form,

$$V(x) \approx (\text{const}) + \frac{1}{2}k(x - x_0)^2 + \mathcal{O}(x - x_0)^3.$$

What is the "force constant" k in terms of $V(x)$ and/or its derivatives?

b) Consider the time-independent Schrödinger equation for a particle with mass m and harmonic oscillator potential $V(x) = \frac{1}{2}kx^2$.

Define $b = \sqrt{\hbar/(m\omega_c)}$. Show that $\psi_0(x) \propto e^{-x^2/2b^2}$ solves the Schrödinger equation for some energy E_0. Using the properties of the normalizable solutions to the Schrödinger equation, explain why $\psi_0(x)$ must describe the ground state.

c) What is the ground state energy E_0? Notice that the quantum mechanical ground state energy is higher than the classical ground state energy. You can think of this as due to the uncertainty principle: if a particle is localized near the bottom of a potential well, then there is an uncertainty in its momentum, and hence its kinetic energy does not vanish.

d) Show that the first excited state is described by the wavefunction

$\psi_1(x) \propto x\, e^{-x^2/2b^2}$ and check that its energy is $E_1 = \frac{3}{2}\hbar\omega_c$. How do you know that this is the first excited state?

One can show (but you don't have to) that the allowed energies are all of the simple form $E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c$, where $\omega_c = \sqrt{k/m}$. You have checked the first two of these.

e) Consider a classical simple harmonic oscillator with energy E_0. Classically the particle moves between two classical turning points, x_1 and x_2 with $x_1 < x_2$. What are x_1 and x_2 in terms of k and m?

f) Write an integral expression for the probablility that the quantum mechanical harmonic oscillator in its ground state will be found outside the classically allowed region, *i.e.* $x < x_1$, or $x > x_2$. You do not need to evaluate the integral.