1. Derivative Interactions

In class we pointed out the subtleties in dealing with derivative interactions. Here you will study a simple example which demonstrates that the naive handling of derivatives gives the correct answer.

Consider a free real scalar field with Lagrangian,

\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{m^2}{2} \phi^2. \]

We can equivalently write this as a Lagrangian in terms of the rescaled field \( \tilde{\phi} \equiv Z^{-1/2} \phi \) as,

\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu} \tilde{\phi})^2 - \frac{m^2}{2} \tilde{\phi}^2 + (Z - 1) \left[ \frac{1}{2} (\partial_{\mu} \tilde{\phi})^2 - \frac{m^2}{2} \tilde{\phi}^2 \right]. \]

Since we know the relation between \( \phi \) and \( \tilde{\phi} \), we also know the relation between two-point functions:

\[ \langle 0 | T (\tilde{\phi}(x)\tilde{\phi}(0)) | 0 \rangle = Z^{-1} \langle 0 | T (\phi(x)\phi(0)) | 0 \rangle. \]

In terms of its Fourier transform, we then have,

\[ \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} \langle 0 | T (\tilde{\phi}(x)\tilde{\phi}(0)) | 0 \rangle = \frac{i Z^{-1}}{k^2 - m^2 + i \epsilon} \]

You are to check this result by evaluating \( \langle 0 | T (\tilde{\phi}(x)\tilde{\phi}(0)) | 0 \rangle \) in perturbation theory, where you are to think of the terms in the Lagrangian proportional to \( (Z - 1) \) as being interaction terms.

a) What are the Feynman rules in this theory, treating the derivatives in the interaction naively?

b) By summing over all diagrams that contribute to the Fourier transformed two-point function, \( \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} \langle 0 | T (\tilde{\phi}(x)\tilde{\phi}(0)) | 0 \rangle \), show that you recover the expected result.
2. Scalar Self Energy

Consider the theory of a single real scalar field,

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{g_3}{3!} \phi^3 - \frac{g_4}{4!} \phi^4. \]

Calculate the one-loop renormalized self energy \( \Pi(k^2) \) for the scalar field \( \phi \). \( \Pi(k^2) \) should satisfy the renormalization conditions \( \Pi(m^2) = 0 \) and \( d\Pi/dk^2|_{k^2=m^2} = 0 \). Your result should be left in terms of integral(s) over a single Feynman parameter.