Physics 722, Spring 2007
Final Exam

Due Friday, May 11, 5pm. Extensions until Wednesday, May 16, 4pm will be allowed.

Rules for the exam: 1) You may use notes you have taken, notes I have given you, and Peskin & Schroeder’s text. That is all you may use, out of fairness to your classmates. 2) You should not (need to) spend more than 9 hours on the exam. If you take an extension due to conflicting schedules, you should not work on the exam over more than a three day period. 3) You should answer problems 1, 2 and 3. Problem 4 is optional.

Feel free to ask questions during the exam. My office phone is 221-3763. My cell phone is 272-2697. I will post answers to questions asked at the course website, http://physics.wm.edu/~erlich/722S07
1. Two-dimensional QED

In this problem you will consider QED in two dimensions, i.e. one space, one time. The Lagrangian looks the same as in QED:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi. \]

The relevant differences from four dimensions are the following:

\[ \int \frac{d^4p}{(2\pi)^4} \rightarrow \int \frac{d^2p}{(2\pi)^2} \]

in loop integrals.

The \( \gamma \)-matrices are 2\( \times \)2 matrices such that \( \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu} \), for example \( \gamma^0 = \sigma_1, \gamma^1 = i\sigma_2 \). They satisfy:

\[ \text{Tr} \gamma^\mu \gamma^\nu = 2g^{\mu\nu}, \]

\[ \text{Tr} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma = 2(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\lambda\nu}). \]

You may also use,

\[ \int d^2p p^\mu F(p^2) = 0, \]

and,

\[ \int d^2p p^\mu p^\nu F(p^2) = \frac{1}{2} g^{\mu\nu} \int d^2p p^2 F(p^2). \]

Other than the changes noted above, the Feynman rules are equivalent to those in 4D QED.

a) Compute the one-loop diagram contributing to the photon self-energy \( i\Pi^{\mu\nu}(k) \), where \( k \) is the (off-shell) photon momentum flowing through the diagram. Show that the loop integral is convergent (or more precisely, the divergent parts cancel), and express it in the form,

\[ i\Pi^{\mu\nu}(k) = g^{\mu\nu} F(k^2) + k^\mu k^\nu G(k^2). \]

You may leave \( F(k^2) \) and \( G(k^2) \) as integrals over a single Feynman parameter. (The remaining integrals should be relatively easy to do.)

b) In 4D, gauge invariance implies the transverseness condition, \( k_\mu \Pi^{\mu\nu}(k) = 0 \). Show that this condition is not satisfied with \( \Pi^{\mu\nu} \) as calculated in part (a). (This is because if you try to prove transverseness from gauge invariance in 2D you encounter linearly divergent integrals that invalidate the proof.)

c) Using the Pauli-Villars regularization procedure, replace the photon self energy \( \Pi^{\mu\nu}(k) \) by the regularized self energy \( \tilde{\Pi}^{\mu\nu}(k, M) = \Pi^{\mu\nu}(k) - \Pi^{\mu\nu}(k, M) \), where in the second term the electron mass is replaced by \( M \).
Show that as $M \to \infty$, $\Pi^{\mu \nu}(k, M)$ remains finite and satisfies the transverseness condition.

2. Ghostly Electromagnetism

We showed in class that gauge fixing conditions of the form $\partial_\mu A^\mu = f(x)$ in QED lead to a Fadeev-Popov determinant independent of the fields in the theory, so that ghosts are not needed for calculations in such gauges. On the other hand, we could be foolish and choose a gauge which requires ghosts in QED. In this problem you will study QED in such a gauge.

Consider the gauge fixing condition $\partial_\mu A^\mu + \sigma A_\mu A^\mu - f(x) = 0$, where $\sigma$ is an arbitrary real number and $f(x)$ is a real-valued function of the spacetime coordinates $x^\mu$.

a) Consider a functional integral of the form

$$\int \mathcal{D}A^\mu \exp \left[ i \int d^4x \mathcal{L} \right] \mathcal{O}(A^\mu),$$

where $\mathcal{O}(A^\mu)$ is a gauge invariant operator.

Introduce a delta function into the functional integral enforcing the gauge-fixing condition, together with the functional determinant which correctly normalizes the delta function.

Integrate over gauge choices for $f(x)$ against the measure

$$\mathcal{D}f(x) \exp \left[ -i \int d^4x \frac{f(x)^2}{2\xi} \right].$$

What effective gauge fixing term appears in the Lagrangian after integrating over functions $f(x)$? We will call this theory QED in Foolish gauge.

b) Now consider the Fadeev-Popov determinant which comes with the gauge-fixing delta function. Show that the determinant can be replaced by an integral over a set of ghost fields with a contribution to the effective Lagrangian, $\mathcal{L}_{\text{ghost}}$. What is $\mathcal{L}_{\text{ghost}}$?

c) What are the Feynman rules for the ghost field propagator and vertices involving the ghost fields?

d) What are the Feynman rules for the photon propagator (which depends on $\xi$), and new vertices involving the photon field in Foolish gauge?
e) Consider the photon self energy $\Pi^{\mu\nu}(k)$ in Foolish gauge with $\xi = 1$. Draw the additional one-loop diagrams contributing to $\Pi^{\mu\nu}(k)$ which are absent in Feynman gauge. Write out the loop integrals for these diagrams using the Foolish gauge Feynman rules, but do not evaluate the integrals (unless you want to).

3. BRST Invariance

Gauge invariance is manifestly lost in the gauge fixed Lagrangian, but a local invariance remains. In this problem you will demonstrate the existence of this local invariance in covariant gauges.

Consider a non-Abelian gauge theory with gauge-fixed Lagrangian

$$L_{gf} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \overline{\Psi}(i\not\!D - m)\Psi - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \overline{c} (-\partial^\mu D_\mu^{ac}) c^c,$$

where $c^a$ and $\overline{c}$ are the ghost and anti-ghost fields. The covariant derivatives are as usual:

$$\not\!D\Psi = \gamma^\mu(\partial_\mu - igA_\mu^a T^a)\Psi$$
$$D_\mu^{ac} c^c = (\partial_\mu \delta_{ac} + gf^{abc} A_\mu^b) c^c,$$

where $T^a$ are the generators of the gauge group in some representation, and $f^{abc}$ are the structure constants of the gauge group.

a) Introduce a set of auxiliary commuting scalar fields $B^a(x)$, and consider the Lagrangian

$$L = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \overline{\Psi}(i\not\!D - m)\Psi + \frac{\xi}{2}(B^a)^2 - B^a \partial^\mu A_\mu^a + \overline{c} (-\partial^\mu D_\mu^{ac}) c^c.$$

There are no kinetic terms for $B^a$ so the field is nondynamical. Performing the Gaussian functional integral over $B^a$ is equivalent to substituting the solution to its equations of motion (i.e. the Euler-Lagrange equation for $B^a$) back into $L$. Show that this procedure leads to $L_{gf}$, demonstrating the equivalence of these theories.

b) Show that $L$ is invariant under the BRST transformation

$$\delta A_\mu^a = \epsilon D_\mu^{ab} c^b,$$
$$\delta \Psi = ig\epsilon c^a T^a \Psi,$$
$$\delta c^a = -\frac{1}{2}g\epsilon f^{abc} B^b c^c,$$
$$\delta \overline{c} = -\epsilon B^a,$$
$$\delta B^a = 0.$$
The parameter $\epsilon$ is a constant anticommuting parameter, i.e. a Grassmann variable that anticommutes with other Grassmann variables. The transformation of $A^a_\mu$ and $\Psi$ looks like a gauge transformation with gauge transformation parameter $\alpha^a(x) = g\epsilon c^a(x)$. You will find the Bianchi identity useful:

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0.$$ 

Comments:
The BRST transformation also has the property of nilpotence, which you are not asked to prove (but it is not difficult). This means that the variation of the variation of any field vanishes. For example, varying the antighost field once gives a term proportional to $B^a$, and varying again, $\delta B^a=0$. The existence of BRST invariance is helpful in proving renormalizability of non-Abelian gauge theories, and for demonstrating that unphysical degrees of freedom (ghosts and extraneous polarizations of the gauge fields) are not produced as final states in interactions.

4. Extra Credit

Quantum chromodynamics (QCD) is an SU(3) gauge theory. Quarks are Dirac fermions which transform in the fundamental representation of SU(3). Show that QCD is confining, i.e. physical particle states are invariant under SU(3) gauge transformations. Show that excitations of the vacuum all have energy separated from the vacuum by at least some finite constant (the mass gap). Collect your $1,000,000 prize from the Clay Institute. Share with your favorite professor.