1. More practice with gamma matrices

Show that:
\[ \gamma^5 \gamma^\mu \gamma^5 = -\gamma^\mu \]
\[ \text{Tr } \phi = 0 \]
\[ \text{Tr } \phi \bar{\psi} = 4a \cdot b \]

Similarly, compute \( \text{Tr } \phi \bar{\psi} \), \( \text{Tr } \phi \bar{\psi} \bar{\phi} \), \( \text{Tr } \phi \bar{\psi} \bar{\phi} \), \( \text{Tr } \phi \gamma^5 \), \( \text{Tr } \phi \bar{\psi} \gamma^5 \), \( \text{Tr } \phi \bar{\psi} \bar{\phi} \gamma^5 \), and \( \text{Tr } \phi \bar{\psi} \bar{\phi} \bar{\phi} \gamma^5 \).

The last of these will involve the constant antisymmetric tensor \( \epsilon^{\mu\nu\rho\sigma} \). Use only the anticommutation relations of the 4x4 gamma matrices. Do not use an explicit representation of the matrices. Recall that \( \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \).

2. Rotations and angular momentum conservation

Consider a system of \( N \) point particles described by the \( N \) vectors \( r^a \), \( a = 1, \ldots, N \), with Lagrangian,
\[ L = \sum_{a=1}^{N} \frac{m_a}{2} |\dot{r}^a|^2 - \sum_{a,b=1}^{N} V_{ab} (|r^a - r^b|) \]

Consider an infinitesimal rotation by angle \( \theta \) about the axis \( e \),
\[ r^a \rightarrow r^a + \theta e \times r^a. \]

a) Show that the action is invariant under rotations.

b) Show that the usual angular momentum is conserved as a consequence of rotation invariance in this system.

3. The Dirac field and electric charge

The Dirac Lagrangian is,
\[ \mathcal{L} = \bar{\psi} (i\partial - m) \psi. \]
a) Treating $\psi$ and $\bar{\psi}$ as independent fields, derive the equations of motion for $\psi$ and $\bar{\psi}$. Show that the equations you derive are self consistent.

b) The transformation $\psi \to e^{i \theta} \psi$, $\bar{\psi} \to e^{-i \theta} \bar{\psi}$ is a symmetry of the theory. What is the associated 4-vector current? What is the charge? (When we quantize the theory we will identify this with the electric charge.)