1. Lorentz transformations

If $\Lambda^\mu_\nu$ describes a Lorentz transformation, such that

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu,$$

then how do the following transform under the Lorentz transformation:

a) The Minkowski tensor, $\eta_{\mu\nu}$?

b) $\partial_\mu \phi(x) \partial^\mu \phi(x)$, where $\phi(x)$ is a scalar field?

c) $\frac{\partial}{\partial x^\nu} \left[ \overline{\psi}(x) \gamma^\mu \psi(x) \right]$, where $\psi(x)$ is a Dirac spinor field?

d) $\overline{\psi}(x) \gamma^\mu \gamma^\nu \psi(x)$?

Prove your results by explicit computation, using the properties of $\Lambda^\mu_\nu$ and the specified representations of the Lorentz group.

2. Dirac spinor representation

The generators of rotations in three dimensions, $T^a$, $a = 1, 2, 3$ satisfy the SO(3) algebra, $[T^a, T^b] = i \epsilon^{abc} T^c$, where $\epsilon^{abc}$ is completely antisymmetric in $a, b$ and $c$, with $\epsilon^{123} = 1$.

The analogous relations for the six generators of Lorentz transformations $J^{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$ with $J^{\mu\nu} = -J^{\nu\mu}$, are,

$$[J^{\mu\nu}, J^{\sigma\rho}] = i \left( g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho} \right).$$

These commutation relations define the Lorentz algebra, SO(3,1). Using the properties of the Dirac $\gamma$-matrices, show that the generators of Lorentz transformations in the Dirac spinor representation,

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu],$$

satisfy the commutation relations describing the Lorentz algebra.
3. Chirality

Any Dirac spinor can be decomposed into a left-handed and a right-handed part by using the chirality projection operators,

\[ P_L = \frac{1 - \gamma^5}{2}, \quad P_R = \frac{1 + \gamma^5}{2}. \]

Using the properties of \( \gamma^5 \) show that:

a) \( P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0. \)

b) Given a Dirac spinor \( \psi \) define its left-handed part as \( \psi_L \equiv P_L \psi \) and its right-handed part as \( \psi_R \equiv P_R \psi \). Show that under a Lorentz transformation of \( \psi \), the left and right-handed components of \( \psi \) transform independently. This implies that the Dirac spinor forms a reducible representation of the Lorentz group.

c) By acting on the Dirac equation \( (i\partial_t - m) \psi = 0 \) with \( P_L \) and with \( P_R \), rewrite the Dirac equation in terms of a coupled set of equations for \( \psi_L \) and \( \psi_R \).

d) Show that the equations for \( \psi_L \) and \( \psi_R \) decouple when \( m \to 0 \).

4. Solution to free Dirac equation with momentum in arbitrary direction

The solution to the Dirac equation for an electron moving in the \( x^3 \)-direction with momentum \( p^3 \) is, in the Weyl basis,

\[ \psi(x) = u(p) e^{ip^\mu x_\mu}, \]

where,

\[ u(p) = \left( \begin{array}{c} \sqrt{E + p^3} \left( \frac{1 - \sigma^3}{2} \right) + \sqrt{E - p^3} \left( \frac{1 + \sigma^3}{2} \right) \xi \\ \sqrt{E + p^3} \left( \frac{1 + \sigma^3}{2} \right) + \sqrt{E - p^3} \left( \frac{1 - \sigma^3}{2} \right) \tilde{\xi} \end{array} \right), \]

and \( \xi \) is a 2-component Pauli spinor.

Show that the above solution can be written as,

\[ u(p) = \left( \begin{array}{c} \sqrt{p^\mu \sigma_\mu} \xi \\ \sqrt{p^\mu \bar{\sigma}_\mu} \tilde{\xi} \end{array} \right), \]

where \( \sigma^\mu = (1, \bar{\sigma}), \quad \bar{\sigma}^\mu = (1, -\bar{\sigma}) \). In this form the solution is valid when the momentum is in an arbitrary direction.
5. *Current conservation for the Dirac equation*

Show that,

\[
\rho = \psi \bar{\psi} \\
\vec{J} = c \psi \bar{\psi} \gamma
\]

satisfy the current conservation equation,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0,
\]

where \( \psi \) satisfies the Dirac equation for an electron coupled to a background electromagnetic field, and \( \gamma \) are the \( 4 \times 4 \) matrices appearing in the Dirac equation.