Phys 475 S'10 Problem Set 4 Solutions

9.5
\[(AAT)^T = (AT)^T AT = AAT \quad \text{since} \quad (AT)^T = A.\]
Since \(AAT\) equals its transpose, it is a symmetric matrix.

9.6
\[
\begin{pmatrix}
2 & 1 \\
1 & 3
\end{pmatrix}
\text{symmetric}
\quad \quad \quad
\begin{pmatrix}
0 & 2 \\
-2 & 0
\end{pmatrix}
\text{skew-symmetric}
\quad
\begin{pmatrix}
1 & 2i \\
3i & 4i
\end{pmatrix}
\text{real}
\quad
\begin{pmatrix}
i & 2i \\
3i & 4i
\end{pmatrix}
\text{pure imaginary}
\]

9.11
\[H^+ = (\overline{H})^T.\]
If \(H\) is real then \(\overline{H} = H\), so \(H^+ = H^T.\)
If \(H\) is Hermitian then \(H^+ = H\). Hence, for a real, Hermitian matrix, \(H^+ = H\), so \(H\) is symmetric.
If \(U\) is unitary then \(U^+U = 1\). If \(U\) is also real then \(U^+U = U^TU = 1\), so \(U\) is an orthogonal matrix.

9.22
If \(U\) is unitary, \([U, U^+] = UU^+ - U^+U = 1 - 1 = 0.\)
If \(O\) is orthogonal, \([O, O^+] = [O, O^T] = O^T - O = 1 - 1 = 0.\)
If \(S\) is symmetric, \([S, S^+] = [S, S^T] = [S, S] = 0.\)
If \(S\) is real, \(A\) is real.
If \(A\) is antisymmetric, \([A, A^+] = [A, AT] = [A, -A] = -[A, A] = 0.\)
If \(H\) is Hermitian, \([H, H^+] = [H, H] = 0.\)
If \(M\) is anti-Hermitian, \([M, M^+] = [M, -M] = -[M, M] = 0.\)

Hence, any matrix belonging to any of these classes of matrices is normal.
9.23
\[(A^{A+})^t = (A^t)^t A^t \geq AA^t \text{ since } (A^t)^t = A. \checkmark \]
\[(A^t + A^t)^t = A^t A = A + A^t. \checkmark \]
\[(i(A-A^t))^t = -i(A^t - A) = i(A-A^t). \checkmark \]
\[\text{imply } i^t = -i \]

7.25
\[a) \text{ Let } A = O^{-1}. \text{ If } O \text{ is orthogonal then } A = O^{-1} = O^T. \]
\[A^T A = (O^T)^T O^T = O O^T = I. \]
Hence, \(A = O^{-1}\) is an orthogonal matrix.
\[b) \text{ Let } A = U^{-1}. \text{ If } U \text{ is unitary then } A = U^{-1} = U^T. \]
\[A^T A = (U^T)^T U^T = U U^T = I. \]
Hence, \(A = U^{-1}\) is a unitary matrix.
\[c) \text{ If } H \text{ is Hermitian and } U \text{ is unitary, then let } A = U^{-1} H U. \]
\[A^T = (U^{-1} H U)^T = U^T H^T (U^T)^T = U^{-1} H U = A \]
\[\text{using } U^{-1} = U^T \text{ using } H^T = H. \]
Hence, \(A = U^{-1} H U\) is a Hermitian matrix.

13.17
\[\{\text{Real numbers \setminus 0}\} \]
\[\checkmark \text{ not inclusively} \]
1) Closure: Product of nonvanishing real numbers is a nonvanishing real number.
2) Ordinary multiplication is associative.
3) Unit element is 1, which is a nonvanishing real number.
4) Inverse of an element is its reciprocal.

Hence, \(\{\text{Real numbers \setminus 0}\}\) w/ ordinary multiplication form a group.

Similarly, for \(\{\text{Complex numbers \setminus 0}\}\).

For \(\{re^{i\theta} \text{ with } r \neq 1\}\), the analysis is also similar. We need to check closure, which is satisfied because the product of two complex numbers with unit magnitude is another complex number with unit magnitude.
1320 If \( \det A = -1 \) and \( \det B = -1 \), then
\[
\det (AB) = (\det A)(\det B) = +1.
\]
Hence, the set of 3x3 matrices w/ determinant -1 is not closed under matrix multiplication.

Furthermore, the identity matrix, which would be the unit element in the group, is not in the set because \( \det 1 = +1 \).
Hence, \( \{ \text{matrices w/ determinant } -1 \} \) with matrix multiplication, orthogonal 3x3 do not form a group.

Comment: In problem 13.17, note that including 0 in the set \( \{ \text{real numbers} \} \) eliminates the existence of an inverse. Since 0 has no inverse. Hence, \( \{ \text{real numbers} \} \) with ordinary multiplication is not a group.